MATH 311/521, FALL 2025 PRACTICE FINAL

DECEMBER 15

Each problem is worth 10 points.

Problem 1. Let $q_e(n)$ and $q_o(n)$ be the number of partitions of n into an even or odd number of distict parts. Give a proof of Euler's identity

$$q_e(n) - q_o(n) = \begin{cases} (-1)^j & n = \frac{3j^2 \pm j}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Hence or otherwise, conclude the formal product identity

$$\phi(x) = \prod_{n=1}^{\infty}(1 - x^n) = 1 + \sum_{j=1}^{\infty}(-1)^j \left(x^{\frac{3j^2+j}{2}} + x^{\frac{3j^2-j}{2}} \right).$$

Problem 2.

a. Define the abscissa of convergence of a Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$.

b. Let $\mu(n)$ be the Möbius function of n, $d(n)$ the number of divisors of n, and $\sigma(n)$ the sum of the divisors of n. Express $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$, $\sum_{n=1}^{\infty} \frac{d(n)}{n^s}$, $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$ as Euler products, and in terms of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, and determine the abscissa of convergence of each series.

c. (Extra credit) Recall that a finite abelian group is isomorphic to a group of form $\bigoplus_p \mathbb{Z}/p^{a_1}\mathbb{Z} \oplus \mathbb{Z}/p^{a_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{a_k}\mathbb{Z}$ where $a_1 \geq a_2 \cdots \geq a_k > 0$. How many isomorphism classes of abelian groups of order p^k exist for p prime? Let $q(n)$ be the number of isomorphism classes of abelian groups of order n. Express $\sum_{n=1}^{\infty} \frac{q(n)}{n^s}$ as an Euler product and in terms of the Riemann zeta function.

Problem 3. The Bernoulli numbers (B_k) are defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$

a. Calculate B_1, B_2, B_3.

b. Take logarithmic derivatives in the product $\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$ to conclude

$$z \cot z = 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} = 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{2k}}{n^{2k}\pi^{2k}}.$$

c. Prove $\zeta(2k) = \frac{2^{2k-1}\pi^{2k}}{(2k)!} B_k$.

Problem 4. Prove a number n is coprime to q if $\sum_{d|\mathrm{GCD}(n,q)} \mu(d) = 1$, and that the sum is zero otherwise. Using this, prove

$$\#\{M \leq n < M + N, \mathrm{GCD}(n, q) = 1\} = \frac{\phi(q)}{q} N + O(2^{\omega(q)})$$

where ϕ is Euler's ϕ function and $\omega(q)$ is the number of distinct primes that divide q.

Problem 5. Let

$$\chi_4(n) = \begin{cases} 1 & n \equiv 1 \bmod 4 \\ -1 & n \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases},$$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $L(s,\chi_4) = \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s}$. The Dedekind zeta function for $\mathbb{Q}(i)$ is

$$\zeta_{\mathbb{Q}(i)}(s) = \zeta(s)L(s,\chi_4) = \sum_{m=1}^{\infty} \frac{r(m)}{m^s}.$$

Prove $r(m) = \#\{x, y \in \mathbb{Z}, x \geq 0, y > 0, x^2 + y^2 = m\}$.

Problem 6. Let $q = p^r$ be a power of a prime and let $\mathbb{F}_q$ be a finite field having q elements.

a. Prove $\mathbb{F}_q$ has characteristic p, that is, if $x \in \mathbb{F}_q$ then $p \cdot x = 0$ for all x, and hence conclude $\mathbb{F}_q$ has $\mathbb{F}_p$ as a subfield.
b. Conclude $\mathbb{F}_q^\times = \{a \in \mathbb{F}_q, a \neq 0\}$ is a multiplicative group, and hence conclude that for all $a \neq 0$, $a^{q-1} = 1$ in $\mathbb{F}_q$. Conclude $x^q - x = \prod_{a \in \mathbb{F}_q}(x - a)$ and that all fields of order q are isomorphic.
c. (Extra credit) For $a \in \mathbb{F}_q^\times$, let $\mathrm{ORD}(a)$ be the least positive n so that $a^n = 1$. For $d|q-1$ let $f(d) = \#\{a \in \mathbb{F}_q^\times : \mathrm{ORD}(a) = d\}$, $g(d) = \#\{a \in \mathbb{F}_q^\times : a^d = 1\}$. Prove $g(d) = \sum_{k|d} f(k)$ and thus $f(d) = \sum_{k|d} \mu(k)\frac{d}{k}$. Conclude $\mathbb{F}_q^\times$ has elements of order $q-1$, and hence is a cycle group.