MATH 311, FALL 2020 MIDTERM 2

OCTOBER 28

Each problem is worth 10 points.

Problem 1.

a. Define the Farey fractions of order n.

b. Given an irrational number ξ, prove that there are infinitely many rationals $\frac{a}{b}$ with $\left|\xi - \frac{a}{b}\right| < \frac{1}{b^2}$.

Solution.

a. The Farey fractions of order n are those reduced fractions $\frac{a}{b}$ with $0 \leq \frac{a}{b} \leq 1$ and $b \leq n$, written in increasing order.

b. Let ξ lie between consecutive Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$ of level n. If $\xi < \frac{a+c}{b+d}$ then $\xi - \frac{a}{b} < \frac{a+c}{b+d} - \frac{a}{b} = \frac{1}{b(b+d)} \leq \frac{1}{b^2}$. If $\xi > \frac{a+c}{b+d}$ then $\frac{c}{d} - \xi \leq \frac{c}{d} - \frac{a+c}{b+d} = \frac{1}{d(b+d)} \leq \frac{1}{d^2}$. Since, by increasing the level of Farey fraction, we can approximate ξ arbitrarily well, there are infinitely many fractions $\frac{a}{b}$ such that $|\frac{a}{b} - \xi| < \frac{1}{b^2}$.

Problem 2. Find all rational points of $x^2 + 5y^2 = 6$ by the secant method.

Solution. $(1,1)$ is on the curve. The secant with rational slope m through $(1,1)$ has equation $y = m(x-1)+1$. This intersects the curve where

$$x^2 + 5(mx - m + 1)^2 - 6 = 0.$$

Dividing by $x - 1$ obtains

$$(5m^2+1)x - 5m^2 + 10m + 1 = 0$$

for the equation of the second point of intersection. This point is thus given by

$$\left(\frac{5m^2 - 10m - 1}{5m^2+1}, \frac{-5m^2 - 2m + 1}{5m^2+1}\right).$$

Problem 3.

a. State the Pigeonhole Principle.
b. Prove that if α is an irrational number, there is a number $1 \leq j \leq n$ such that $j\alpha$ has distance at most $\frac{1}{n}$ from the nearest integer.

Solution.

a. If S and T are finite sets with $|S| > |T|$ and $f : S \to T$ then there is $t \in T$ such that $|f^{-1}(t)| > 1$.
b. Consider $0\alpha, 1\alpha, 2\alpha, ..., n\alpha$ mod 1. Since there are $n+1$ of these numbers, two fall into one of the intervals
$$\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), ..., \left[\frac{n-1}{n}, 1\right).$$
Say $i\alpha, j\alpha$ fall into the same interval modulo 1 with $i < j$. Then $(j-i)\alpha$ falls into the interval $\left[-\frac{1}{n}, \frac{1}{n}\right]$ modulo 1, which satisfies the requirements.

Problem 4. Find all integer solutions to the given system.

$$\begin{aligned} 5w + 3x + 2y + z &= 9 \\ w + x + y + z &= 2 \\ 2w + 7x + y + 3z &= 4. \end{aligned}$$

Solution. Make the reductions

$$\begin{array}{ccccc} 5 & 3 & 2 & 1 & 9 \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 7 & 1 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array} \leftrightarrow \begin{array}{ccccc} 4 & 2 & 1 & 0 & 7 \\ 1 & 1 & 1 & 1 & 2 \\ -1 & 4 & -2 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}$$

$$\leftrightarrow \begin{array}{ccccc} 4 & 2 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 7 & 8 & 0 & 0 & 12 \\ \hline 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ -1 & -1 & -1 & 1 & \end{array}$$

$$\leftrightarrow \begin{array}{ccccc} 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 7 & 8 & 0 & 0 & 12 \\ \hline 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ -4 & -2 & 1 & 0 & \\ 3 & 1 & -1 & 1 & \end{array}$$

$$\leftrightarrow \begin{array}{ccccc} 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 7 & 1 & 0 & 0 & 12 \\ \hline 1 & -1 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ -4 & 2 & 1 & 0 & \\ 3 & -2 & -1 & 1 & \end{array}$$

$$\leftrightarrow \begin{array}{ccccc} 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 12 \\ \hline 8 & -1 & 0 & 0 & \\ -7 & 1 & 0 & 0 & \\ -18 & 2 & 1 & 0 & \\ 17 & -2 & -1 & 1 & \end{array}$$

The solution is thus

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 & -1 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ -18 & 2 & 1 & 0 \\ 17 & -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} t \\ 12 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} -12+8t \\ 12-7t \\ 31-18t \\ -29+17t \end{pmatrix}.$$