MATH 307, FALL 2020 PRACTICE MIDTERM 2 SOLUTIONS

OCTOBER 26

Each problem is worth 10 points.

Problem 1. Find all critical points of $f(x, y) = x^4 - x^2 + y^2$ and determine if each is a local min, a local max or a saddle point.

Solution. We have

$$\nabla f(x, y) = \begin{pmatrix} 4x^3 - 2x \\ 2y \end{pmatrix}.$$

Thus the critical points are $(0, 0)$ and $(\pm\frac{1}{\sqrt{2}}, 0)$. The Hessian is $\begin{pmatrix} 12x^2 - 2 & 0 \\ 0 & 2 \end{pmatrix}$. Thus $(0, 0)$ is a saddle point and the other critical points are local minima.

Problem 2.

a. Maximize $x^3 + 2y^3$ on $\{x^2 + y^2 \leq 1\}$.
b. Maximize $x^3 + y$ on $\{x^2 + y^2 = 1\}$.

Solution.

a. Let $f(x, y) = x^3 + 2y^3$ so that $\nabla f(x, y) = \begin{pmatrix} 3x^2 \\ 6y^2 \end{pmatrix}$. Thus the only critical point is at $(0, 0)$ with value 0, but this is neither a max nor min as there are positive and negative points in any neighborhood of $(0, 0)$. Thus the max or min appears on $\{x^2 + y^2 = 1\}$. By Lagrange multipliers, an extreme point satisfies

$$\begin{pmatrix} 3x^2 \\ 6y^2 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

or $\left(\frac{x}{y}\right)^2 = 2\left(\frac{x}{y}\right)$ or $x = 0$ or $y = 0$. This obtains the potential solutions $(0, \pm 1), (\pm 1, 0)$, $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, $(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$. The maximum is 2 at $(0, 1)$.

b. By Lagrange multipliers,

$$\begin{pmatrix} 3x^2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus either $x = 0$ and $y = \pm 1$ or $3xy = 1$. In the latter case $(x+y)^2 = \frac{5}{3}$, $(x-y)^2 = \frac{1}{3}$. The maximum is evidently obtained with x, y positive, so we may assume $(x + y) = \sqrt{\frac{5}{3}}$. This obtains the candidates

$$x = \frac{\sqrt{5} \pm 1}{2\sqrt{3}}, \qquad y = \frac{\sqrt{5} \mp 1}{2\sqrt{3}}.$$

The maximum among these candidates is $\left(\frac{\sqrt{5}+1}{2\sqrt{3}}, \frac{\sqrt{5}-1}{2\sqrt{3}}\right)$ with value $\frac{1+5\sqrt{5}}{6\sqrt{3}}$.

Problem 3. Determine if the function

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous and differentiable at 0.

Solution. If $x^2+y^2 = \delta > 0$, then $|x|^3 \leq \delta^{\frac{3}{2}}$ and $|y|^3 \leq \delta^{\frac{3}{2}}$ so $\left|\frac{x^3-y^3}{x^2+y^2}\right| \leq 2\delta^{\frac{1}{2}}$, which tends to 0 as $\delta \to 0$. Thus f is continuous at 0.

We will check that f is not differentiable at 0. Its matrix of partial derivatives is given by $\begin{pmatrix} 1 & -1 \end{pmatrix}$. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a unit vector. If f were differentiable, the directional derivative in direction u would be $u_1 - u_2$. However,

$$\lim_{t\to 0} \frac{f(tu) - f(0)}{t} = \lim_{t\to 0} u_1^3 - u_2^3.$$

This is a contradiction.

Problem 4. Let $f(x,y,z) = \begin{pmatrix} e^{xyz} \\ xy \end{pmatrix}$, and $g(u,v) = u^2 + v^2$. Calculate f', g' and $(g \circ f)'$.

Solution. We have

$$f'(x,y,z) = \begin{pmatrix} yze^{xyz} & xze^{xyz} & xye^{xyz} \\ y & x & 0 \end{pmatrix}, \qquad g'(u,v) = \begin{pmatrix} 2u & 0 \\ 0 & 2v \end{pmatrix}.$$

Thus

$$\begin{aligned}(g \circ f)'(x,y,z) &= g'(f(x,y,z))f'(x,y,z) \\ &= \begin{pmatrix} 2e^{xyz} & 0 \\ 0 & 2xy \end{pmatrix} \begin{pmatrix} yze^{xyz} & xze^{xyz} & xye^{xyz} \\ y & x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2yze^{2xyz} & 2xze^{2xyz} & 2xye^{2xyz} \\ 2xy^2 & 2x^2y & 0 \end{pmatrix}.\end{aligned}$$

Problem 5. Let $F(u,v) = \begin{pmatrix} u^3 - v^3 \\ 3u^2 v \end{pmatrix}$. Find $F'(1,2)$ and $(F^{-1})'(-7,6)$.

Solution. We have

$$F'(u,v) = \begin{pmatrix} 3u^2 & -3v^2 \\ 6uv & 3u^2 \end{pmatrix}, \qquad F'(1,2) = \begin{pmatrix} 3 & -12 \\ 12 & 3 \end{pmatrix}.$$

By the inverse function theorem, since F' is non-singular, F has a local inverse at $F(1,2) = (-7,6)$. The derivative satisfies

$$(F^{-1})'(-7,6) = (F'(1,2))^{-1} = \frac{1}{153} \begin{pmatrix} 3 & 12 \\ -12 & 3 \end{pmatrix}.$$

Problem 6. Find the volume of the largest rectangular solid with sides parallel to the coordinate planes, which fits inside $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$.

Solution. This is equivalent to maximizing $8xyz$ subject to $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$. Substitute $x_1 = 3x$, $y_1 = 2y$, $z_1 = z$ so that this becomes maximize $48x_1y_1z_1$ subject to $x_1^2 + y_1^2 + z_1^2 = 1$. By Lagrange multipliers, the optimum is attained where

$$\begin{pmatrix} y_1z_1 \\ x_1z_1 \\ x_1y_1 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

We may assume $x_1, y_1, z_1 \geq 0$, and thus all three must be equal to obtain the optimum, thus all equal to $\frac{1}{\sqrt{3}}$. The optimum is thus $\frac{16}{\sqrt{3}}$.