MATH 307, FALL 2020 PRACTICE FINAL SOLUTION

DECEMBER 9

Each problem is worth 10 points.

Problem 1. Determine the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

Solution. Let $\lambda' + 1 = \lambda$. The characteristic polynomial is

$$P(\lambda') = \det \begin{pmatrix} -\lambda' & 2 & 0 \\ 2 & 2-\lambda' & -1 \\ 0 & -1 & -\lambda' \end{pmatrix} = -\lambda'(\lambda'^2 - 2\lambda' - 5).$$

Thus the eigenvalues are $\lambda' = 0$ and $\lambda' = 1 \pm \sqrt{6}$ or $\lambda = 1$ and $\lambda = 2 \pm \sqrt{6}$.

The eigenvalue corresponding to $\lambda = 1$ is in the null space of $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ and hence is a multiple of $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. The eigenvector of $\lambda = 2 \pm \sqrt{6}$ is in the null space of $\begin{pmatrix} -1 \mp \sqrt{6} & 2 & 0 \\ 2 & 1 \mp \sqrt{6} & -1 \\ 0 & -1 & -1 \mp \sqrt{6} \end{pmatrix}$. Thus v is a multiple of $\begin{pmatrix} \frac{2}{-1\mp\sqrt{6}} \\ 1 \\ \frac{-1}{-1\mp\sqrt{6}} \end{pmatrix}$.

Problem 2.

a. Calculate a potential function for $\mathbb{F} = \begin{pmatrix} \frac{-y}{x^2+y^2} + yze^{xyz} \\ \frac{x}{x^2+y^2} + xze^{xyz} \\ xye^{xyz} + 2z \end{pmatrix}$.

b. Calculate $\operatorname{div} F$.

c. Let $\gamma(t) = \begin{pmatrix} 1 \\ t^2 \\ t^{10} \end{pmatrix}$ for $0 \le t \le 1$. Calculate $\int_\gamma \mathbb{F} \cdot dx$.

Solution.

a. $f(x,y,z) = \arctan\left(\frac{y}{x}\right) + e^{xyz} + z^2$.

b. $\operatorname{div} \mathbb{F} = (x^2y^2 + x^2z^2 + y^2z^2)e^{xyz} + 2$.

c. Since the field is conservative this is $f(1,1,1) - f(1,0,0) = \arctan(1) + e + 1 - \arctan(0) - 1 = \frac{\pi}{4} + e$.

Problem 3.

a. Let $\mathbb{F}(x,y,z) = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}$. Calculate $\operatorname{div}\mathbb{F}$ and $\operatorname{curl}\mathbb{F}$.

b. Determine an outward pointing normal vector N to the surface $S = \{x^2 + y^2 + z^2 = 1\}$ and calculate

$$\int_S \mathbb{F} \cdot N d\sigma.$$

Solution.

a. $\operatorname{div}\mathbb{F} = 3x^2 + 3y^2 + 3z^2$, $\operatorname{curl}\mathbb{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

b. The gradient of $x^2 + y^2 + z^2$ is in the direction $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, which is a unit vector.

c. Let B be the unit ball. By the divergence theorem

$$\begin{aligned} \int_S \mathbb{F} \cdot N d\sigma &= \int_B \operatorname{div}\mathbb{F} dV \\ &= \int_{x^2+y^2+z^2 \leq 1} 3(x^2 + y^2 + z^2) dV \\ &= 3 \int_0^1 \rho^4 d\rho \int_0^\pi \sin\phi d\phi \int_0^{2\pi} d\theta \\ &= \frac{12\pi}{5}. \end{aligned}$$

Problem 4. Calculate the outward flux through the surface of the cylinder $C = \{(x, y, z) : x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$ of the field $\mathbb{F}(x, y, z) = \begin{pmatrix} x \\ y \\ e^{xy} \end{pmatrix}$.

Solution. The unit normals of the top and bottom of the cylinder point in the opposite z direction, so these integrals cancel. This leaves the flux through the curved part of the cylinder. Here the unit normal points radially in the xy plane, and $F \cdot N = 1$. Thus the total flux is 4π.

Problem 5.

a. Given the curve $\gamma(t) = \begin{pmatrix} t \\ t^2 \\ \frac{2}{3}t^3 \end{pmatrix}$. Calculate the unit tangent vector $T(t)$, the principal normal vector $N(t)$ and the binormal $B(t)$.

b. Find the length of the curve between $0 \leq t \leq 1$.

Solution.

a. We have $\gamma'(t) = \begin{pmatrix} 1 \\ 2t \\ 2t^2 \end{pmatrix}$, so $\|\gamma'\| = \sqrt{1 + 4t^2 + 4t^4} = (2t^2 + 1)$. Thus

$$T(t) = \frac{1}{2t^2 + 1} \begin{pmatrix} 1 \\ 2t \\ 2t^2 \end{pmatrix}.$$

Differentiating,

$$T'(t) = \frac{1}{(2t^2 + 1)^2} \begin{pmatrix} -4t \\ -4t^2 + 2 \\ 4t \end{pmatrix}$$

and hence

$$N(t) = \frac{1}{2t^2 + 1} \begin{pmatrix} -2t \\ -2t^2 + 1 \\ 2t \end{pmatrix}.$$

We have

$$\begin{aligned} B(t) &= T(t) \times N(t) \\ &= \frac{1}{(2t^2 + 1)^2} \det \begin{pmatrix} i & j & k \\ 1 & 2t & 2t^2 \\ -2t & -2t^2 + 1 & 2t \end{pmatrix} \\ &= \frac{1}{2t^2 + 1} \begin{pmatrix} 2t^2 \\ -2t \\ 1 \end{pmatrix}. \end{aligned}$$

b. The arc length is

$$\int_0^1 \|\gamma'(t)\| dt = \int_0^1 2t^2 + 1 dt = \frac{5}{3}.$$

Problem 6. The distance $y(t)$ covered by a falling body of mass m in time t subject to atmospheric resistance satisfies

$$\frac{d^2y}{dt^2} + \frac{k}{m}\frac{dy}{dt} = g$$

where g is the gravitational constant and k is a friction coefficient.

a. Show that the law of motion satisfies

$$y(t) = c_1 + c_2 e^{-\frac{kt}{m}} + \frac{mg}{k}t.$$

b. Determine c_1 and c_2 such that $y(0) = y_0$, $y'(0) = v_0$.

Solution.

a. The system has particular solution $y_p = \frac{mg}{k}t$. The homogeneous system is $(D + \frac{k}{m})Dy = 0$, which has homogeneous solution $c_1 + c_2 e^{-\frac{k}{m}t}$. This obtains the solution.

b. We have $y(0) = c_1 + c_2 = y_0$, $y'(0) = \frac{mg}{k} - \frac{k}{m}c_2 = v_0$, so

$$c_2 = \frac{m}{k}\left(\frac{mg}{k} - v_0\right), \qquad c_1 = y_0 - c_2.$$

Problem 7. Find the closest point to $(1, 2)$ of the ellipse

$$x^2 + 4y^2 = 1.$$

Solution. The distance function squared is $(x-1)^2 + (y-2)^2$. By Lagrange multipliers, the optimal point satisfies

$$\begin{pmatrix} x-1 \\ y-2 \end{pmatrix} = \lambda \begin{pmatrix} x \\ 4y \end{pmatrix}$$

or $4y(x-1) = x(y-2)$ or $3xy + 2x - 4y = 0$. Let $u = x + 2y$, $v = x - 2y$. We have

$$(x^2 + 4xy + 4y^2) + \frac{8}{3}(x - 2y) = u^2 + \frac{8}{3}v = 1$$
$$(x^2 - 4xy + 4y^2) - \frac{8}{3}(x - 2y) = v^2 - \frac{8}{3}v = 1.$$

The latter equation has solutions $v = -\frac{1}{3}$ and $v = 3$. Since $u^2 + v^2 = 2$ on the ellipse, we conclude $v = -\frac{1}{3}$. Thus $u = \pm\frac{\sqrt{17}}{3}$. We have $x = \frac{u+v}{2} = \frac{\pm\sqrt{17}-1}{6}$, $y = \frac{u-v}{4} = \frac{\pm\sqrt{17}+1}{12}$. The minimum occurs at the positive solution, $\left(\frac{\sqrt{17}-1}{6}, \frac{\sqrt{17}+1}{12}\right)$.

Problem 8. Find the tangent plane and a normal vector to the surface $x^2 + 2y^2 - z^2 = 2$ at $(1, 1, 1)$.

Solution. The gradient of the level curve is $\begin{pmatrix} 2x \\ 4y \\ -2z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$. The equation of the tangent plane is $2(x-1)+4(y-1)-2(z-1) = 0$ or $2x+4y-2z = 4$.

Problem 9. Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ and $G : \mathbb{R}^4 \to \mathbb{R}^2$ be given by

$$F(x, y, z) = \begin{pmatrix} xy \\ yz \\ zx \\ x^2 + y^2 + z^2 \end{pmatrix}, \qquad G(s, t, u, v) = \begin{pmatrix} s^2 + t^2 \\ u^2 - v^2 \end{pmatrix}.$$

Calculate F', G' and $(G \circ F)'$.

Solution. We have

$$F' = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \\ 2x & 2y & 2z \end{pmatrix}.$$

We have

$$G' = \begin{pmatrix} 2s & 2t & 0 & 0 \\ 0 & 0 & 2u & -2v \end{pmatrix}.$$

We have, by the chain rule,

$$(G \circ F)'(x, y, z) = G'(F(x, y, z))F'(x, y, z)$$

$$= \begin{pmatrix} 2xy & 2yz & 0 & 0 \\ 0 & 0 & 2zx & -2(x^2 + y^2 + z^2) \end{pmatrix} \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \\ 2x & 2y & 2z \end{pmatrix}$$

$$= \begin{pmatrix} 2xy^2 & 2x^2y + 2yz^2 & 2y^2z \\ 2z^2x - 4x(x^2 + y^2 + z^2) & -4y(x^2 + y^2 + z^2) & 2zx^2 - 4z(x^2 + y^2 + z^2) \end{pmatrix}.$$