MATH 307, FALL 2020 PRACTICE FINAL

DECEMBER 9

Each problem is worth 10 points.

Problem 1. Determine the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

Problem 2.

a. Calculate a potential function for $\mathbb{F} = \begin{pmatrix} \frac{-y}{x^2+y^2} + yze^{xyz} \\ \frac{x}{x^2+y^2} + xze^{xyz} \\ xye^{xyz} + 2z \end{pmatrix}$.

b. Calculate $\operatorname{div} F$.

c. Let $\gamma(t) = \begin{pmatrix} 1 \\ t^2 \\ t^{10} \end{pmatrix}$ for $0 \leq t \leq 1$. Calculate $\int_\gamma \mathbb{F} \cdot dx$.

Problem 3.

a. Let $\mathbb{F}(x,y,z) = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}$. Calculate $\operatorname{div} \mathbb{F}$ and $\operatorname{curl} \mathbb{F}$.

b. Determine an outward pointing normal vector N to the surface $S = \{x^2 + y^2 + z^2 = 1\}$ and calculate

$$\int_S \mathbb{F} \cdot N d\sigma.$$

Problem 4. Calculate the outward flux through the surface of the cylinder $C = \{(x, y, z) : x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$ of the field $\mathbb{F}(x, y, z) = \begin{pmatrix} x \\ y \\ e^{xy} \end{pmatrix}$.

Problem 5.

a. Given the curve $\gamma(t) = \begin{pmatrix} t \\ t^2 \\ \frac{2}{3}t^3 \end{pmatrix}$. Calculate the unit tangent vector $T(t)$, the principal normal vector $N(t)$ and the binormal $B(t)$.

b. Find the length of the curve between $0 \leq t \leq 1$.

Problem 6. The distance $y(t)$ covered by a falling body of mass m in time t subject to atmospheric resistance satisfies

$$\frac{d^2y}{dt^2} + \frac{k}{m}\frac{dy}{dt} = g$$

where g is the gravitational constant and k is a friction coefficient.

a. Show that the law of motion satisfies

$$y(t) = c_1 + c_2 e^{-\frac{kt}{m}} + \frac{mg}{k}t.$$

b. Determine c_1 and c_2 such that $y(0) = y_0$, $y'(0) = v_0$.

Problem 7. Find the closest point to $(1,2)$ of the ellipse

$$x^2 + 4y^2 = 1.$$

Problem 8. Find the tangent plane and a normal vector to the surface $x^2 + 2y^2 - z^2 = 2$ at $(1, 1, 1)$.

Problem 9. Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ and $G : \mathbb{R}^4 \to \mathbb{R}^2$ be given by

$$F(x, y, z) = \begin{pmatrix} xy \\ yz \\ zx \\ x^2 + y^2 + z^2 \end{pmatrix}, \qquad G(s, t, u, v) = \begin{pmatrix} s^2 + t^2 \\ u^2 - v^2 \end{pmatrix}.$$

Calculate F', G' and $(G \circ F)'$.