MATH 307, FALL 2020 MIDTERM 1 SOLUTIONS

SEPTEMBER 28

Each problem is worth 10 points.

Problem 1.

a. Determine the distance between $\begin{pmatrix}1\\2\end{pmatrix}$ and the line

$$\ell(t) = \begin{pmatrix}1\\-1\end{pmatrix} + t\begin{pmatrix}4\\3\end{pmatrix}.$$

b. Determine the distance between the point $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and the plane $2x + 4y + z = 0$.

Solution.

a. This is equal to the distance between $\begin{pmatrix}0\\3\end{pmatrix}$ and the line through 0, $\begin{pmatrix}4t\\3t\end{pmatrix}$. The a unit vector orthogonal to the line is $u = \begin{pmatrix}\frac{-3}{5}\\\frac{4}{5}\end{pmatrix}$, and hence the distance to the line is $\begin{pmatrix}0\\3\end{pmatrix} \cdot u = \frac{12}{5}$.

b. A unit vector orthogonal to the plane is given by $u = \frac{1}{\sqrt{21}}\begin{pmatrix}2\\4\\1\end{pmatrix}$, and hence the distance to the plane is

$$\left|\begin{pmatrix}1\\0\\-1\end{pmatrix} \cdot u\right| = \frac{1}{\sqrt{21}}.$$

Problem 2.

a. Find the length of the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and the angle between them.

b. Calculate the angle between $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and the plane $2x + 3y - z = 0$.

Solution.

a. Let $u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Then $|u| = \sqrt{2}$, $|v| = \sqrt{5}$ and $u \cdot v = 2$. The angle between them satisfies $\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{2}{\sqrt{10}}$.

b. Let $w = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ which is perpendicular to the plane. Thus the angle with the plane is $\frac{\pi}{2} - \theta$, where θ is the angle between $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$. We have

$$\theta = \cos^{-1} \frac{8}{\sqrt{70}}.$$

Problem 3.

a. Calculate

$$\det \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 3 & 1 \end{pmatrix}.$$

b. Calculate the inverse matrix of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

a. Subtract the third column from the first, then the first from the second, and second from fourth to conclude

$$\begin{aligned} \det \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 3 & 1 \end{pmatrix} &= \det \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} = -2. \end{aligned}$$

b. By row reduction

$$\begin{pmatrix}1&2&3\\4&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}1&2&0\\4&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&-3\\0&1&0\\0&0&1\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&0\\0&-7&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&-3\\-4&1&12\\0&0&1\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&-3\\\frac{4}{7}&-\frac{1}{7}&-\frac{12}{7}\\0&0&1\end{pmatrix}$$

$$\to\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}-\frac{1}{7}&\frac{2}{7}&\frac{3}{7}\\\frac{4}{7}&-\frac{1}{7}&-\frac{12}{7}\\0&0&1\end{pmatrix}.$$

The last matrix is the inverse.

Problem 4. Calculate the area of the triangle with vertices $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Solution. Two sides of the triangle have vectors $u = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $v = \begin{pmatrix}3\\0\\0\end{pmatrix}$.

The area is $\frac{1}{2}|u \times v|$. Notices that $u \times v = w \times v$ where $w = \begin{pmatrix}0\\1\\1\end{pmatrix}$ with w, v orthogonal, so the area is $\frac{1}{2}|v||w| = \frac{3}{\sqrt{2}}$.

Problem 5.

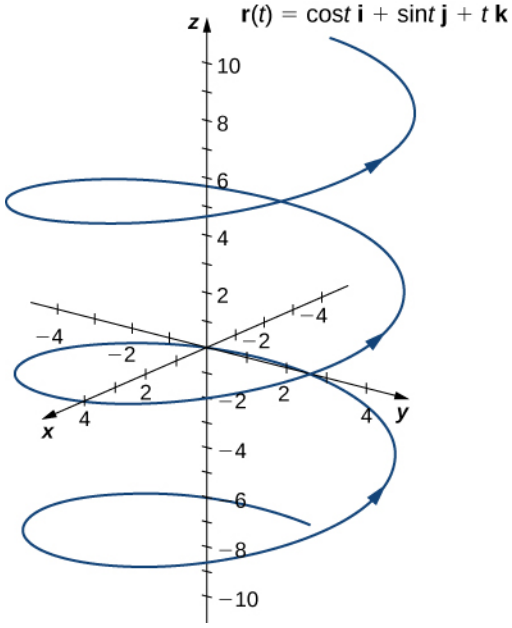
a. Find the tangent line to $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$ at $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

b. Sketch the curve between $0 \leq t \leq 2\pi$.

Solution.

a. We have $\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$ which, at $t = 0$ has $\gamma'(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. The tangent line is thus $\begin{pmatrix} 1 \\ t \\ t \end{pmatrix}$ with $t \in \mathbb{R}$.

b. The graph is the segment with $0 \leq t \leq 2\pi$ of the picture



Problem 6.

a. Determine the velocity and acceleration of the trajectory

$$x(t) = \begin{pmatrix} t^2 \\ 2t \\ 2 \end{pmatrix}.$$

b. Find the closest points of this curve to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Solution.

a. $v(t) = \begin{pmatrix} 2t \\ 2 \\ 0 \end{pmatrix}, a(t) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

b. This distance is $\sqrt{t^4 + 4t^2 + 4}$, which is minimized at $t = 0$. Thus the closest point is $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.