RECALL FROM LAST LECTURE:

<u>THEOREM:</u> LET $\{\underline{x}_1, \ldots, \underline{x}_n\}$ BE VECTORS SPANNING A VECTOR SPACE $\mathcal{V}$. THEN THERE IS A BASIS FOR $\mathcal{V}$ WHICH IS A SUBSET OF $\{\underline{x}_1, \ldots, \underline{x}_n\}$.

THEOREM: LET $\dim(\mathcal{V}) = n$.

ANY TWO OF FOLLOWING IMPLIES THE 3^{rd}.

(1) A SET $\mathcal{B}$ CONTAINS n VECTORS

(2) $\mathcal{B}$ IS LINEARLY INDEP.

(3) $\mathcal{B}$ SPANS.

THEOREM (RANK-NULLITY THM):

LET V, W BE FINITE D'ML VECTOR SPACES, AND

$f: V \to W$ LINEAR.

LET R BE A REDUCED FORM FOR THE MATRIX OF f W.R.T. TWO BASES.

THE DIMENSION OF THE IMAGE IS r, WHERE r IS THE NUMBER OF LEADING VARIABLES, THE NULL SPACE HAS DIMENSION k, WHERE k IS THE NUMBER OF NON-PIVOT

NULLITY = DIM(NULL(f))

RANK = DIM(IMAGE)

NULLITY + RANK = DIM(DOMAIN)

LET $B = \text{rref}(A)$

"ROW REDUCED ECHELON FORM."

$$B = \begin{bmatrix} 1 & * & * & 0 & * & & & * \\ 0 & 0 & 0 & 1 & * & \cdots & & * \\ & & & 0 & 0 & 0 & 0 & \\ & 0 & & & & & 1 & * & * \\ \hline 0 & 0 & 0 & & & & & 0 \end{bmatrix}$$

THE SPAN OF THE COLUMNS OF B IS CONTAINED IN THE SPAN OF THE PIVOT COLUMNS. THERE ARE r OF THESE, LINEARLY INDEPENDENT.

IF $B = C \cdot A$, WITH C INVERTIBLE, THE RANGE OF THE MAP IS SPANNED BY

$B\underline{e}_{\alpha_1}, \ldots, B\underline{e}_{\alpha_r}$, $\alpha_1, \ldots, \alpha_r$ LOCATION OF PIVOTS.

THESE ARE THE VECTORS

$B\underline{e}_{\alpha_1} = \underline{f}_1, \ldots, B\underline{e}_{\alpha_r} = \underline{f}_r$

THE RANGE $B\underline{x}$ IS CONTAINED IN THE SPAN OF $\underline{f}_1, \ldots, \underline{f}_r$, WHICH ARE LINEARLY INDEP.

$A\underline{e}_{\alpha_1}, \ldots, A\underline{e}_{\alpha_r} = C^{-1}B\underline{e}_{\alpha_1}, \ldots, C^{-1}B\underline{e}_{\alpha_r}$
$= C^{-1}f_1, \ldots, C^{-1}f_r.$

AND $A\underline{x} = C^{-1}B\underline{x}$,

$C^{-1}\underline{f}_1, \ldots, C^{-1}\underline{f}_r$ ARE LINEARLY INDEPENDENT, SINCE ANY COMBINATION ADDING TO $\underline{0}$ OBTAINS A RELATION BETWEEN THE $\underline{f}_i$ BY APPLYING C.

SIMILARLY, IF

$B\underline{x} = u_1\underline{f}_1 + \cdots + u_r\underline{f}_r$

THEN $A\underline{x} = C^{-1}B\underline{x} =$
$u_1C^{-1}\underline{f}_1 + \cdots + u_rC^{-1}\underline{f}_r$

SO $C^{-1}\underline{f}_1, \ldots, C^{-1}\underline{f}_r$ SPAN $\Rightarrow$ A BASIS.

THUS, DIM (RANGE(A)) = r.

EIGENVECTORS AND EIGENVALUES:

LET $L: V \to V$, V A VECTOR SPACE.

AN EIGENVECTOR $\underline{u} \in V$ SOLVES THE EQUATION $L \cdot \underline{u} = \lambda \cdot \underline{u}$

λ IS A SCALAR, THE EIGENVALUE.

IF THE $\underline{u}_1, \ldots, \underline{u}_k$ FORM A <u>BASIS</u> THEN W.R.T. THIS BASIS, L HAS MATRIX

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_k \end{pmatrix}$$ DIAGONAL.

<u>THEOREM</u>: THE MATRIX OF AN OPERATOR L WITH RESPECT TO A BASIS $\{\underline{u}_1, \ldots, \underline{u}_k\}$ OF EIGENVECTORS IS DIAGONAL, WITH iTH DIAGONAL ENTRY THE iTH EIGENVALUE.

PROOF: BY INDUCTION ON k.

(BASE CASE)

IF $k=1$, $\{\underline{u}_1\}$ IS LINEARLY INDEP.

(INDUCTIVE STEP)

SUPPOSE $\{\underline{u}_1, \ldots, \underline{u}_{k-1}\}$ ARE LIN. INDEP.

SUPPOSE FOR CONTRADICTION THAT $\{\underline{u}_1, \ldots, \underline{u}_k\}$ ARE DEPENDENT.

THEN $\underline{u}_k = c_1 \underline{u}_1 + \ldots + c_{k-1} \underline{u}_{k-1}$.

$$(L - \lambda_k I)\underline{u}_k = L\underline{u}_k - \lambda_k \underline{u}_k = \underline{0}.$$

$$\underline{0} = (L - \lambda_k I)(c_1 \underline{u}_1 + \cdots + c_{k-1}\underline{u}_{k-1})$$

$$\underline{0} = c_1(\lambda_1 - \lambda_k)\underline{u}_1 + c_2(\lambda_2 - \lambda_k)\underline{u}_2 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)\underline{u}_{k-1}$$

BY LINEAR INDEPENDENCE,

$$c_i(\lambda_i - \lambda_k) = 0 \Rightarrow c_i = 0$$

ALL i, A CONTRADICTION. $\square$

<u>THEOREM</u>: LET L BE A LINEAR OPERATOR ON $\mathbb{R}^n$, AND SUPPOSE

$$\det(A-\lambda I)=0$$

HAS n DISTINCT ROOTS $\lambda_1,\dots,\lambda_n$.

THEN THE CORRESPONDING EIGENVECTORS $\underline{u}_1,\dots,\underline{u}_n$ FORM A BASIS.

<u>PROOF</u>: THIS FOLLOWS SINCE $\underline{u}_1,\dots,\underline{u}_n$ ARE LINEARLY INDEPENDENT. $\square$

DEFINE $U = \begin{pmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{pmatrix}$.

$U\underline{y} = \underline{x}$.

IF A IS THE MATRIX OF A LINEAR MAP $f: \mathbb{R}^n \to \mathbb{R}^n$, WITH RESPECT TO THE STANDARD BASIS THEN $B = U^{-1}AU$

IS THE MATRIX OF f W.R.T. COORDINATES ON $\underline{u}_1, \ldots, \underline{u}_n$.

RECALL THAT THE DOT PRODUCT ON $\mathbb{R}^n$ SATISFIES

(1) POSITIVITY: $\underline{x} \cdot \underline{x} \geq 0$
$= 0$ IFF $\underline{x} = \underline{0}$

(2) SYMMETRIC: $\underline{x} \cdot \underline{y} = \underline{y} \cdot \underline{x}$.

(3) BILINEAR:
$(a\underline{x} + b\underline{y}) \cdot \underline{z} = a \cdot \underline{x} \cdot \underline{z} + b \cdot \underline{y} \cdot \underline{z}$.

(ALSO THE 2^{ND} VARIABLES BY SYMMETRY)

THE LENGTH OR NORM OF $\underline{x}$ IS $\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$.

THIS GENERALIZES THE EUCLIDEAN LENGTH.

PROOF: IF EITHER $\underline{x} = \underline{0}$ OR $\underline{y} = \underline{0}$ THE THEOREM HOLDS WITH EQUALITY, SO ASSUME $\underline{x} \neq \underline{0}$, $\underline{y} \neq \underline{0}$.

$$0 \le \|\underline{x} - t\underline{y}\|^2 = \langle \underline{x} - t\underline{y}, \underline{x} - t\underline{y} \rangle$$
$$= \langle \underline{x}, \underline{x} \rangle - t\langle \underline{x}, \underline{y} \rangle - t\langle \underline{y}, \underline{x} \rangle + t^2 \langle \underline{y}, \underline{y} \rangle$$
$$= \|\underline{x}\|^2 - 2t\langle \underline{x}, \underline{y} \rangle + t^2 \cdot \|\underline{y}\|^2.$$

THIS IS MINIMIZED AT $t = \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{y}\|^2}$. ITS VAUE THERE IS $\|\underline{x}\|^2 - 2\frac{\langle \underline{x}, \underline{y} \rangle^2}{\|\underline{y}\|^2} + \frac{\langle \underline{x}, \underline{y} \rangle^2}{\|\underline{y}\|^2}$

HENCE $0 \le \|\underline{x}\|^2 - \frac{\langle \underline{x}, \underline{y} \rangle^2}{\|\underline{y}\|^2}$

$\iff \langle \underline{x}, \underline{y} \rangle^2 \le \|\underline{x}\|^2 \|\underline{y}\|^2$

SO $|\langle \underline{x}, \underline{y} \rangle| \le \|\underline{x}\| \cdot \|\underline{y}\|$.

EQUALITY HOLDS IFF $\underline{x} - t\underline{y} = \underline{0}$ HAS A SOL'N $\iff$ $\underline{x}, \underline{y}$ DEPENDENT. $\square$

PROOF:

(1) $\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} \geq 0$

$= 0$ IFF $\underline{x} = \underline{0}$.

(2) $\|t \cdot \underline{x}\| = \sqrt{\langle t\underline{x}, t\underline{x} \rangle} = \sqrt{t^2 \cdot \langle \underline{x}, \underline{x} \rangle}$

$= |t| \sqrt{\langle \underline{x}, \underline{x} \rangle}$

(3) FOR THE TRIANGLE INEQUALITY

$\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

$\iff \|\underline{x} + \underline{y}\|^2 \leq (\|\underline{x}\| + \|\underline{y}\|)^2$

$\iff \langle \underline{x} + \underline{y}, \underline{x} + \underline{y} \rangle \leq \|\underline{x}\|^2 + \|\underline{y}\|^2 + 2\|\underline{x}\|\|\underline{y}\|.$

$\iff \cancel{\langle \underline{x}, \underline{x} \rangle} + \langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{x} \rangle + \cancel{\langle \underline{y}, \underline{y} \rangle} \leq \cancel{\|\underline{x}\|^2} + \cancel{\|\underline{y}\|^2} + 2\|\underline{x}\|\|\underline{y}\|.$

$\iff 2\langle \underline{x}, \underline{y} \rangle \leq 2\|\underline{x}\|\|\underline{y}\|.$

THIS FOLLOWS BY CAUCHY-SCHWARZ.

$\square$