MAT 307: LECTURE 8

EIGEN VECTORS AND
EIGENVALUE

MIDTERM 1 NEXT MONDAY

<u>THEOREM</u>: LET $\mathcal{V}$ BE A VECTOR SPACE, AND LET $\{\underline{x}_1, \ldots, \underline{x}_n\}$ BE A LINEARLY INDEPENDENT SET IN $\mathcal{V}$. THEN EITHER THERE IS A BASIS CONTAINING $\{\underline{x}_1, \ldots, \underline{x}_n\}$ OR $\mathcal{V}$ IS ∞-DIM'L, AND $\underline{x}_1, \ldots, \underline{x}_n$ IS PART OF AN INFINITE SEQ. OF L. I. VECTORS.

PROOF: IF (2) AND (3) HOLD THEN $\mathcal{B}$ IS A BASIS, SO $|\mathcal{B}| = \dim V = n \Rightarrow (1)$.

IF (1) AND (2) HOLD, ADD 0 OR MORE VECTORS TO $\mathcal{B}$ TO OBTAIN A BASIS, WITH n ELEMENTS. SINCE NO ELEMENTS ARE ADDED $\mathcal{B}$ ITSELF SPANS.

IF (1) AND (3) HOLD, CHOOSE A SPANNING SUBSET OF $\mathcal{B}$ WHICH IS A BASIS, HENCE HAS n ELEMENTS. THIS PROVES $\mathcal{B}$ IS LINEARLY INDEPENDENT. $\square$

PROOF: RECALL THAT AFTER CHOOSING BASES

$\{\underline{v}_1, \dots, \underline{v}_n\}$ FOR $\mathcal{V}$

$\{\underline{w}_1, \dots, \underline{w}_m\}$ FOR $\mathcal{W}$

$f: \mathcal{V} \to \mathcal{W}$ IS GIVEN BY MATRIX MULTIPLICATION

$$A \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \leftrightarrow f(v_1 \underline{v}_1 + \dots + v_n \underline{v}_n)$$

$$= w_1 \underline{w}_1 + \dots + w_m \underline{w}_m.$$

SIMILARLY,

$$\dim(\text{null}(B)) = \dim(\text{null}(A))$$

SINCE $B = C \cdot A$

$\Leftrightarrow B\underline{x} = \underline{0} \Leftrightarrow CA\underline{x} = 0$

$\Leftrightarrow C^{-1}B\underline{x} = \underline{0} \Leftrightarrow A\underline{x} = 0.$

THE DIMENSION OF THE NULL SPACE OF B IS EQUAL TO THE NUMBER OF NON-PIVOT VARIABLES

$$\begin{pmatrix} 1 * * & 0 & 0 & \\ 0 & 1 * * & 0 & \\ 0 & 0 & 1 & * \\ & & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = 0.$$

SINCE IF $\underline{B}_1, \dots, \underline{B}_k$ ARE THE NON-PIVOT COLUMNS, FOR ANY CHOICE $b_1\underline{B}_1 + \dots + b_k\underline{B}_k$

THERE IS A <u>UNIQUE</u> CHOICE OF PIVOT VARIABLES SO THAT THE LINEAR COMBINATION IS 0. SINCE THE NULL SPACE IS UNIQUELY DETERMINED BY k PARAMETERS, ITS DIMENSION IS k.

$\square$

IF THERE ARE SEVERAL EIGENVECTORS $\underline{u}_1, \dots, \underline{u}_k$,

$$L\underline{u}_i = \lambda_i \underline{u}_i$$

THEN $$L\left(\sum_{i=1}^{k} c_i \underline{u}_i\right) = \sum_{i=1}^{k} c_i \lambda_i \underline{u}_i .$$

WE HAVE

$$\det\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix} = \lambda_1 \lambda_2 \cdots \lambda_k .$$

SIMILARLY, THE CHARACTERISTIC EQUATION IS $\det(\lambda I - L)$

$$= \det\begin{pmatrix} \lambda-\lambda_1 & & & \\ & \lambda-\lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda-\lambda_k \end{pmatrix} = (\lambda-\lambda_1)\cdots(\lambda-\lambda_k).$$

$\lambda_1, \ldots, \lambda_k$ ARE CHARACTERISTIC ROOTS.

THEOREM: LET $\underline{u}_1, \dots, \underline{u}_k$ BE (NON-ZERO) EIGENVECTORS WITH DISTINCT EIGENVALUES. THEN $\{\underline{u}_1, \dots, \underline{u}_k\}$ ARE LINEARLY INDEP.

SUPPOSE

$\det(A-\lambda I)=0.$

THIS MEANS THAT $A-\lambda I$ IS NOT INVERTIBLE. THUS THERE IS $x \neq 0$ WITH $(A-\lambda I)x = 0$

OR $Ax = \lambda x$ IS AN EIGENVECTOR.

THEOREM: FOR EACH ROOT λ OF $\det(A-\lambda I)=0$, THERE IS AN EIGENVECTOR W/ E.V. λ.

CHANGE OF BASIS MATRICES.

LET $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ BE THE COORDINATES OF $\mathbb{R}^n$ W.R.T. THE STANDARD BASIS VECTORS.

LET $\underline{u}_1, \ldots, \underline{u}_n$ BE A BASIS.

LET $y_1 \underline{u}_1 + \ldots + y_n \underline{u}_n = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\Longleftrightarrow \begin{pmatrix} | & | & & | \\ \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

PROOF:

$\underline{z} = A\underline{x}$ IN STANDARD COORDINATES.

IN $\underline{u}_1, \dots, \underline{u}_n$ COORDINATES THE COORDINATES OF z ARE

$U^{-1}A\underline{x} = U^{-1}AU\underline{y}$. $\square$

(NOTE: U IS INVERTIBLE, SINCE $n \times n$ AND COLUMNS L.I.)

DEFINITION: AN INNER PRODUCT $\langle \cdot , \cdot \rangle$ ON A VECTOR SPACE V SATISFIES THE FOLLOWING:

(1) POSITIVITY: $\langle \underline{x}, \underline{x} \rangle \geq 0$,
$= 0$ IFF $\underline{x} = \underline{0}$.

(2) SYMMETRY: $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$

(3) BILINEAR: $\langle a\underline{x} + b\underline{y}, \underline{z} \rangle$
$= a\langle \underline{x}, \underline{z} \rangle + b\langle \underline{y}, \underline{z} \rangle$.

CAUCHY-SCHWARZ INEQUALITY:

$$|\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \cdot \|\underline{y}\|$$

WITH EQUALITY IFF $\underline{x}, \underline{y}$ ARE LINEARLY INDEPENDENT.

THE <u>NORM</u> SATISFIES:

(1) POSITIVITY: $\|\underline{x}\| \geq 0$, EQUAL IFF $\underline{x} = \underline{0}$.

(2) HOMOGENEOUS: $\|t \cdot \underline{x}\| = |t| \cdot \|\underline{x}\|$ FOR $t \in \mathbb{R}$

(3) TRIANGLE INEQUALITY: $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$.