MIDTERM 1, IN CLASS,
IS A WEEK FROM MONDAY.
COVERS ALL MATERIAL THRU
NEXT WEDNESDAY'S CLASS

<u>DEFINITION</u>: A FUNCTION f IS A <u>LINEAR FUNCTION</u> IF ITS DOMAIN AND RANGE ARE VECTOR SPACES AND

$$\boxed{f(s\underline{x}+t\underline{y}) = s f(\underline{x}) + t f(\underline{y})}$$

WHERE $\underline{x}, \underline{y}$ ARE VECTORS AND $s, t \in \mathbb{R}$ ARE SCALARS.

PROOF: $f(\underline{x}) = f(\underline{y})$

$\Longleftrightarrow f(\underline{x} - \underline{y}) = f(\underline{x}) - f(\underline{y})$

$= \underline{0}$. $\square$

<u>THEOREM:</u> IF $f : \mathcal{U} \to \mathcal{V}$ AND $g : \mathcal{V} \to \mathcal{W}$ ARE LINEAR MAPS THEN

$$g \circ f : \mathcal{U} \to \mathcal{W}$$

IS LINEAR.

<u>THEOREM</u>: IF $f: V \to W$ AND $g: V \to W$ ARE LINEAR, SO IS $af + bg$.

<u>NOTE</u>: THE SPACE OF ALL LINEAR MAPS $V \to W$ IS WRITTEN $\mathcal{L}(V, W)$. IT IS ITSELF A VECTOR SPACE.

<u>THEOREM:</u> IF $f: \mathcal{V} \to \mathcal{W}$ IS LINEAR THEN $f(\mathcal{V}) = \{f(v) : v \in \mathcal{V}\}$ IS A SUBSPACE. IF ITS 1-1, THEN $f^{-1}: f(\mathcal{V}) \to \mathcal{V}$ IS LINEAR.

THEOREM: SUPPOSE $f: \mathcal{V} \to \mathcal{W}$ IS LINEAR, AND $\mathcal{U} \subset \mathcal{V}$ IS A SUBSPACE. THEN $f(\mathcal{U}) \subset \mathcal{W}$ IS A SUBSPACE.

PROOF: SINCE $\mathcal{U} \subset \mathcal{V}$ IS A SUBSPACE, $\mathcal{U}$ ITSELF IS A VECTOR SPACE.

$f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{W}$ IS LINEAR, SO ITS IMAGE $f|_{\mathcal{U}}(\mathcal{U}) = f(\mathcal{U})$ IS A SUBSPACE. $\square$

EXAMPLE: GIVEN THE LINEAR

$$f\begin{pmatrix} x \\ y \end{pmatrix} = x + 2y$$

$$\text{NULL}(f) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x = -2y \right\}$$

$$= \left\{ \begin{pmatrix} -2y \\ y \end{pmatrix} : y \in \mathbb{R} \right\}.$$

PROOF: IF $\underline{u}, \underline{v} \in \text{NULL}(f)$,

THEN

$$f(a\underline{u}+b\underline{v}) = a\,\overset{0}{\cancel{f(\underline{u})}} + b\,\overset{0}{\cancel{f(\underline{v})}}$$

$$= \underline{0}$$

SO $a\underline{u}+b\underline{v} \in \text{NULL}(f)$. $\square$

RECALL:

DEFINITION: GIVEN $S \subset V$, V A VECTOR SPACE,

$$\text{SPAN}(S) = \{ c_1 \underline{s}_1 + c_2 \underline{s}_2 + \dots + c_n \underline{s}_n : n \in \mathbb{N}, c_1, \dots, c_n \in \mathbb{R}, \underline{s}_j \in S \}$$

= SET OF FINITE LINEAR COMBINATIONS FROM S.

<u>DEFINITION</u>: A SET $S \subset V$ IS <u>LINEARLY INDEPENDENT</u> IF NO VECTOR IN S IS A LINEAR COMBINATION OF OTHER VECTORS FROM S.

<u>DEFINITION:</u> IF V HAS A FINITE BASIS $\{\underline{b}_1, \ldots, \underline{b}_n\}$ THEN $\dim(V) = n$.

IF V IS <u>NOT</u> SPANNED BY ANY FINITE LIST OF VECTORS IT IS INFINITE DIMENSIONAL.

THEOREM CT'D: IF

$$\underline{u} = x_1 \underline{b}_1 + \dots + x_n \underline{b}_n \longleftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$\underline{v} = y_1 \underline{b}_1 + \dots + y_n \underline{b}_n \longleftrightarrow \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

THEN $a\underline{u} + \underline{v} = (ax_1 + y_1)\underline{b}_1 + \dots + (ax_n + y_n)\underline{b}_n$

$$a \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

THEOREM: LET V HAVE BASIS $\{\underline{v}_1, \dots, \underline{v}_n\}$ AND W HAVE BASIS $\underline{w}_1, \dots, \underline{w}_m$. LET $f: V \to W$ BE LINEAR.

SUPPOSE

$$f(\underline{v}_j) = a_{1j}\underline{w}_1 + a_{2j}\underline{w}_2 + \dots + a_{mj}\underline{w}_j.$$

THEN

$$f(v_1\underline{v}_1 + \dots + v_n\underline{v}_n) = w_1\underline{w}_1 + \dots + w_m\underline{w}_m$$

WHERE THE COORDS ARE RELATED BY

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{m \times n} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}.$$

[ONCE BASES ARE CHOSEN FOR TWO VECTOR SPACES, A LINEAR MAP IS JUST MATRIX MULT IN THE COORDINATES.]

EXAMPLE: DIFFERENTIATION d

MAPS $P_3 = \{$POLYS, DEG $\leq 3\}$

TO ITSELF.

WITH BASIS $\{1, x, x^2, x^3\}$

$d(1) = 0$ $d(x^2) = 2x$

$d(x) = 1$ $d(x^3) = 3x^2$

d HAS MATRIX

$$\boxed{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

NULL(d) = SPAN(1).

PROOF: LET $\underline{v}_1, \dots, \underline{v}_n$ BE A BASIS, AND $\underline{x}_1, \dots, \underline{x}_m$ m VECTORS WITH $m > n$.

LET $\underline{x}_i = a_{1i}\underline{v}_1 + a_{2i}\underline{v}_2 + \dots + a_{ni}\underline{v}_i$ IN COORDINATES.

THEN THE LINEAR COMBINATION $r_1\underline{x}_1 + r_2\underline{x}_2 + \dots + r_m\underline{x}_m$ HAS COORDINATES

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} ?$$

$m > n$.

THE EQUATION $A \cdot \underline{r} = \underline{0}$ ALWAYS HAS A NON-TRIVIAL SOLUTION SINCE $m > n$; THAT IS, WHEN

A IS PUT IN ROW REDUCED ECHELON FORM, THERE IS ALWAYS A NON-PIVOT COLUMN

WHICH IS A LINEAR COMB. OF THE OTHER COLUMNS. $\square$

<u>PROOF:</u> IF $\{\underline{w}_1, \ldots, \underline{w}_m\}$ ANOTHER BASIS, $m < n$ THEN $\underline{v}_1, \ldots, \underline{v}_n$ ARE LINEARLY DEP. IF $m > n$, $\underline{w}_1, \ldots, \underline{w}_m$ ARE LIN. DEP. EITHER IS A CONTRADICTION. $\square$

PROOF: EITHER $\underline{v}_1, \dots, \underline{v}_n$ ARE L.I., AND THUS A BASIS, OR SOME $\underline{v}_j$, SAY W.L.O.G. $\underline{v}_n$, IS A COMBINATION OF $\underline{v}_1, \dots, \underline{v}_{n-1}$.

SAY $\underline{v}_n = c_1\underline{v}_1 + \dots + c_{n-1}\underline{v}_{n-1}$.

THEN THE SPAN OF $\{\underline{v}_1, \dots, \underline{v}_{n-1}\}$ IS STILL V, SINCE IF $\underline{x} \in V$,

$$\begin{aligned}\underline{x} &= a_1\underline{v}_1 + \dots + a_n\underline{v}_n \\ &= a_1\underline{v}_1 + \dots + a_{n-1}\underline{v}_{n-1} + a_n(c_1\underline{v}_1 + \dots + c_{n-1}\underline{v}_{n-1}) \\ &= (a_1 + a_nc_1)\underline{v}_1 + \dots + (a_{n-1} + a_nc_{n-1})\underline{v}_{n-1}\end{aligned}$$

IS IN SPAN$\{\underline{v}_1, \dots, \underline{v}_{n-1}\}$.

CONTINUE DISCARDING VECTORS UNTIL NO LONGER POSSIBLE. THE REMAINING SET STILL SPANS AND IS LINEARLY INDEPENDENT. $\square$

PROOF: SUPPOSE S_1 IS L.I. AND DOES NOT SPAN V. THEN CHOOSE $\underline{x} \in V$, $\underline{x} \notin \mathrm{SPAN}(S_1)$.

CLAIM $S_1 \cup \{\underline{x}\}$ IS STILL L.I.

IF $c_1\underline{s}_1 + \dots + c_n\underline{s}_n + c\underline{x} = \underline{0}$

WHERE $\underline{s}_1, \dots, \underline{s}_n$ DISTINCT ELEMS OF S_1, THEN EITHER

$c = 0 \Rightarrow c_1, \dots, c_n = 0$ BY L.I.

$c \neq 0 \rightarrow \underline{x} = -\frac{1}{c}(c_1\underline{s}_1 + \dots + c_n\underline{s}_n)$
$\in \mathrm{SPAN}(S_1)$, CONTRADICTION.

THUS $S_1 \cup \{\underline{x}\}$ IS L.I.

CONTINUE ADDING VECTORS THIS WAY. IF AT SOME POINT CANNOT ADD ANOTHER VECTOR THEN L.I. AND SPAN $\Rightarrow$ BASIS.

OTHERWISE INFINITE L.I. LIST. $\square$