MAT 307 LECTURE 5

INVERSES, DETERMINANTS.

<u>EXAMPLE:</u> IF $ad-bc \neq 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

<u>THEOREM:</u> IF A IS INVERTIBLE, THEN $A\underline{x} = \underline{b}$ HAS THE UNIQUE SOLUTION

$$\underline{x} = A^{-1}\underline{b}.$$

EXAMPLE:

$$x + 2y = 3$$
$$3x + 7y = -4 \iff \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

$$\iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 29 \\ -13 \end{pmatrix}.$$

$\square$

EXAMPLE:

$$A \quad \begin{pmatrix} 2 & 4 & 8 \\ 1 & 0 & 0 \\ 1 & -3 & -7 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[r_3 - r_2]{r_1 - 2r_2} \begin{pmatrix} 0 & 4 & 8 \\ 1 & 0 & 0 \\ 0 & -3 & -7 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{r_1 \to r_1/4} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & -3 & -7 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{r_3 + 3r_1} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ \frac{3}{4} & -\frac{5}{2} & 1 \end{pmatrix}$$

$$\xrightarrow[-r_3]{r_1 + 2r_3} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{7}{4} & -\frac{11}{2} & 2 \\ 0 & 1 & 0 \\ -\frac{3}{4} & \frac{5}{2} & -1 \end{pmatrix}$$

$$\xrightarrow[r_1, r_2]{\text{SWAP}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \boxed{\begin{pmatrix} 0 & 1 & 0 \\ \frac{7}{4} & -\frac{11}{2} & 2 \\ -\frac{3}{4} & \frac{5}{2} & -1 \end{pmatrix}}$$

$$= A^{-1} ☆$$

PROOF: SUPPOSE A IS INVERTIBLE.

THEN $A\underline{x} = \underline{0} \iff \underline{x} = A^{-1}\cdot\underline{0}$
$= \underline{0}$.

SO ONLY ONE SOLUTION.

ON THE OTHER HAND,

LET B BE A RREF (ROW REDUCED ECHELON FORM OF A). THEN $A\underline{x} = \underline{0} \iff B\underline{x} = \underline{0}$

$B\underline{x} = \underline{0}$.

$$B = \begin{pmatrix} 0 \cdots 0 & 1 & * & * \\ 0 & \cdots 0 & 0 \cdots 0 \; 1 & \vdots \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \\ 0 \; 0 & \cdots & \cdots & 0 \\ 0 & \cdots & & 0 \end{pmatrix}_{n\times n}$$

k PIVOTS.

$n-k$ ROWS OF 0'S.

IF $k<n$ THEN THE COLUMNS OF B ARE LINEARLY DEPENDENT

BECAUSE THE PIVOT COLUMNS ARE $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_k \right\} = \{\underline{e}_1, \ldots, \underline{e}_k\}$.

ANY NON-PIVOT COLUMN OF THE MATRIX (ONE EXISTS) HAS FORM $\begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = x_1\underline{e}_1 + \cdots + x_k\underline{e}_k$

SO ONE OF THE NON-PIVOT COLUMNS IS A LINEAR COMBINATION OF THE PIVOT COLUMNS.

THIS MEANS THAT THERE IS A NON-ZERO SOLUTION

$B\underline{x} = \underline{0} \iff A\underline{x} = \underline{0}$, $\underline{x} \neq 0$. $\square$

<u>THEOREM</u>: IF $A_{n\times n}$ IS INVERTIBLE THE SIDE-BY-SIDE ROW REDUCTION PROCESS ENDS IN $\left(I_{n\times n} \mid A^{-1}\right)$.

AS CONSTRUCTED IN THE PROOF, ANY INVERTIBLE MATRIX A IS THE PRODUCT OF ELEMENTARY MATRICES.

LET A BE AN $n \times n$ SQUARE MATRIX, E.G.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

SUM OVER ALL CHOICES OF ONE ENTRY PER ROW/COLUMN TAKING THEIR PRODUCT, TOGETHER WITH A SIGN CORRESPONDING TO THE ORDERING OF THE ENTRIES.

E.G.

$$f: \begin{tikzcd} 1 \arrow[d] & 2 \arrow[dr] & 3 \arrow[dl] & 4 \arrow[d] \\ 1 & 2 & 3 & 4 \end{tikzcd}$$

1 INVERSION

THE SIGN OF THE PERMUTATION IS $(-1)^{\#\text{INVERSIONS}}$

WHERE EACH INVERSION IS A CROSSED ARROW AS ABOVE

ROW EXPANSION FORMULA:

LET A_{ij} "ij TH MINOR" FOUND BY DELETING ROW i, COLUMN j FROM A.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

THIS IS EXPANSION BY THE iTH ROW.

<u>THEOREM</u>: THE DETERMINANT $\det A$ IS LINEAR AS A FUNCTION OF EACH ROW OR COLUMN SEPARATELY.

$$\det\begin{pmatrix} - & \underline{\alpha}_1 & - \\ & \vdots & \\ - & \underline{\alpha}_{i-1} & - \\ - & a\underline{\beta}_i + b\underline{\gamma}_i & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix} = a\cdot\det\begin{pmatrix} - & \underline{\alpha}_1 & - \\ & \vdots & \\ - & \underline{\beta}_i & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix} + b\cdot\det\begin{pmatrix} - & \underline{\alpha}_1 & - \\ & \vdots & \\ - & \underline{\gamma}_i & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix}.$$

<u>THEOREM</u>: IF B IS OBTAINED BY SWAPPING TWO ROWS OF A, THEN

$$\det(B) = -\det A.$$

LET $n > 2$ AND ASSUME TRUE FOR $(n-1)\times(n-1)$ MATRICES.

GIVEN A WHICH IS $n\times n$ LET A' BE OBTAINED BY SWAPPING TWO ROWS.

COROLLARY: IF A HAS TWO EQUAL ROWS, $\det(A)=0$.

PROOF: SWAPPING THEM LEAVES A UNCHANGED, BUT SWITCHES THE SIGN OF $\det A$

So $\det A=0$.