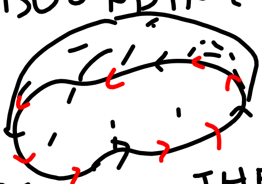
STOKES THEOREM:

SURFACE S, ORIENTED IN $\mathbb{R}^3$, ∂S IS A BOUNDING CURVE γ



THE THEOREM RELATES THE LINE INTEGRAL OF A VECTOR FIELD F AROUND γ WITH THE FLUX INTEGRAL OF curl F THROUGH THE SURFACE.

NOTE: div(curl(F))

$= \nabla \cdot (\nabla \times F)$, SCALAR TRIPLE PRODUCT

$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$,

$$= \det \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0.$$

THIS MEANS THAT

$$\text{curl}(F) = \det \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

IS A DIVERGENCE-FREE FIELD, AND HENCE SATISFIES THE INDEPENDENCE OF SURFACE PROPERTY BY GAUSS'S THEOREM.

PROOF: WE'LL GIVE THE PROOF FOR $\mathbb{F} = \begin{pmatrix} F_1 \\ 0 \\ 0 \end{pmatrix}$ THE EXTENSION TO A GENERAL FIELD HOLDS BY LINEARITY.

$$\text{curl}\,\mathbb{F} = \begin{pmatrix} 0 \\ \frac{\partial F_1}{\partial z} \\ -\frac{\partial F_1}{\partial y} \end{pmatrix}$$

$$\left[\det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{pmatrix} = -\underline{j} \cdot \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & 0 \end{pmatrix} + \underline{k} \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & 0 \end{pmatrix} \right]$$

THUS THE STATEMENT WHICH WE WISH TO PROVE IS

$$\boxed{\oint_{\partial S} F_1\,dx = \int_S -\frac{\partial F_1}{\partial y}\,dx\,dy + \frac{\partial F_1}{\partial z}\,dz\,dx.}$$

(2)

WE HAVE:

THE DIFFERENTIAL dx IS GIVEN BY

$$\frac{d}{dt}\left[g_1(h(t))\right] dt$$
$$= \left(\frac{\partial g_1}{\partial u} \cdot \frac{du}{dt} + \frac{\partial g_1}{\partial v} \cdot \frac{dv}{dt}\right) dt.$$

THUS

$$\int_{\partial S} F_1 \cdot dx = \int F_1(g(h(t))) \left(\frac{\partial g_1}{\partial u}\frac{du}{dt} + \frac{\partial g_1}{\partial v}\frac{dv}{dt}\right) dt$$

$$= \oint_{S=\partial D} F_1(g) \cdot \frac{\partial g}{\partial u} du + \oint_{S=\partial D} F_1(g) \cdot \frac{\partial g}{\partial v} dv.$$

THIS EXPRESSES THE ORIGINAL LINE INTEGRAL OF F_1 ABOUT ∂S AS A LINE INTEGRAL ABOUT ∂D IN THE PARAMETERIZING SPACE. THIS IS A 2-DIM'L LINE INTEGRAL.

BY GREEN'S THEOREM FOR D, δ_1

$$\oint_{\partial S_1} F_1\, dx = \iint_D \left[\frac{\partial}{\partial u}\left(F_1(g)\frac{\partial g_1}{\partial v}\right) - \frac{\partial}{\partial v}\left(F_1(g)\frac{\partial g_1}{\partial u}\right)\right] du\,dv.$$

[GREENS' THM: $\oint_{\partial D} F\,dx + G\,dy = \int_D \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\, dA$]

PRODUCT RULE.

$$\frac{\partial}{\partial u}\left[F_1(g)\cdot\frac{\partial g_1}{\partial v}\right] = \left[\frac{\partial F_1}{\partial x}\cdot\frac{\partial g_1}{\partial u} + \frac{\partial F_1}{\partial y}\cdot\frac{\partial g_2}{\partial u} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial g_3}{\partial u}\right]\cdot\frac{\partial g_1}{\partial v}$$

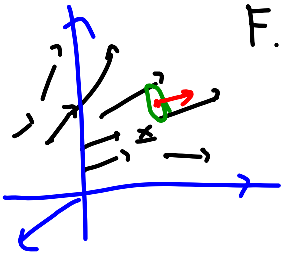
$$F_1\begin{pmatrix} g_1(u,v) \\ g_2(u,v) \\ g_3(u,v)\end{pmatrix}$$

$$+ F_1(g)\cdot\frac{\partial^2 g_1}{\partial u\,\partial v}$$

$$\frac{\partial}{\partial v}\left(F_1(g)\frac{\partial g_1}{\partial u}\right) = \left[\frac{\partial F_1}{\partial x}\frac{\partial g_1}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial g_2}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial g_3}{\partial v}\right]\frac{\partial g_1}{\partial u}$$

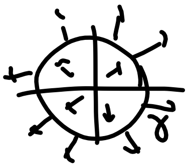
$$+ F_1(g)\frac{\partial^2 g_1}{\partial u\,\partial v}$$

INTERPRETATION:



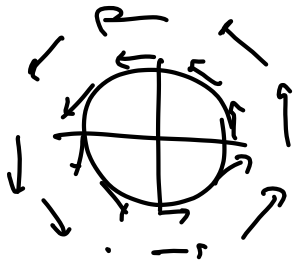
F. Take a small disc D about a point $\underline{x}$.

The integral $\int_{\partial D} F \cdot dx$ describes how much the field F rotates about the axis perpendicular to the disk.



Radial field $\underline{F} \cdot T = 0$

No rotation.



$\underline{F} \cdot \underline{T} = 1$.

Field rotates.

curl F is essentially constant at x by continuity, so the surface integral

$$\int_D \text{curl } F \cdot \underline{N} \, d\sigma \simeq \text{area disc} \times \text{curl } F \cdot \underline{N}$$

$$= \int_{\partial D} F \cdot dx.$$

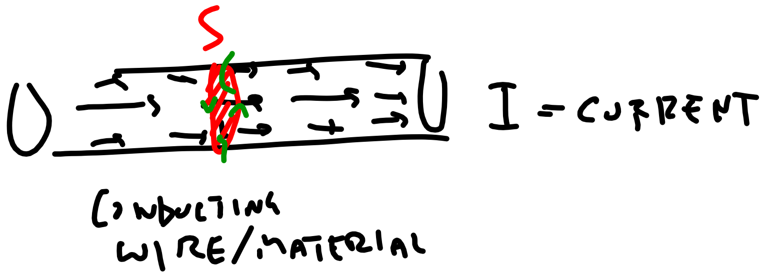
Thus this is maximized if the disk's normal vector is pointed in the direction of curl F.

PHYSICAL INTERPRETATION:

ACCORDING TO MAXWELL'S EQUATIONS THE CURRENT FLOW I IS RELATED TO A MAGNETIC FIELD B BY

$$\text{curl } B = I.$$

THUS



THE FLUX OF CURRENT PASSING THROUGH SOME CROSS-SECTIONAL SURFACE S IS

$$\int_S I \cdot dS = \int_S \text{curl } B \cdot dS$$

$$= \int_{\partial S} B \cdot dx$$

THE LINE INTEGRAL OF THE MAGNETIC FIELD AROUND THE BOUNDARY.

THIS IS CALLED: AMPÈRE'S LAW

EXAMPLE: IF $G(x,y,z)$

$= (y\sin z,\ x\cos z,\ z\sin x)$

THEN $F = \text{curl}\, G$ IS DIVERGENCE-FREE.

$$F = \det \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z & x\cos z & z\sin x \end{vmatrix}$$

$$= \underline{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\cos z & z\sin x \end{vmatrix} - \underline{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y\sin z & z\sin x \end{vmatrix}$$

$$+ \underline{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y\sin z & x\cos z \end{vmatrix}$$

$$= \underline{i}\left(x\sin z\right) + \underline{j}\left(-z\cos x + y\cos z\right)$$

$$+ \underline{k}\left(\cos z - \sin z\right).$$

THIS IS AN EXAMPLE OF A DIVERGENCE FREE FIELD.

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}.$$

SATISFIES THE FOLLOWING:

(1) FOR SCALARS a, b

$$\nabla(af+bg) = a\nabla f + b\cdot\nabla g.$$

(2) $\nabla(f\cdot g) = \nabla f\cdot g + f\cdot\nabla g.$

(3) $\nabla\times(af+bg) = a\nabla\times f + b\nabla\times g.$

(4) $\nabla\times(\underbrace{f}_{\text{SCALAR FN}}\cdot\underbrace{\mathbb{F}}_{\text{FIELD}}) = f\cdot\nabla\times\mathbb{F} + \nabla f\cdot\mathbb{F}.$

(5) $\nabla\cdot(a\cdot\mathbb{F} + b\cdot\mathbb{G}) = a\,\nabla\cdot\mathbb{F} + b\,\nabla\cdot\mathbb{G}$

(6) $\nabla\cdot(\underbrace{f}_{\text{SCALAR FN}}\cdot\underbrace{\mathbb{F}}_{\text{FIELD}}) = f\cdot\nabla\cdot\mathbb{F} + \nabla f\cdot\mathbb{F}.$

(7) $\nabla\cdot(\mathbb{F}\times\mathbb{G}) = (\nabla\times\mathbb{F})\cdot\mathbb{G} - (\nabla\times\mathbb{G})\cdot\mathbb{F}.$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$$

IS CALLED THE LAPLACIAN.

IT PLAYS AN IMPORTANT ROLE IN PHYSICS BECAUSE IT IS TRANSLATION AND ROTATION INVARIANT.

ITS USES: • DESCRIBE THE DISSIPATION OF HEAT
• THE BEHAVIOR OF A WAVE IN A SOLID BODY

• APPEARS IN SCHÖDINGER'S EQUATON FROM QUANTUM MECHANICS.

WE CAN ARGUE INSTEAD WITH

$f \nabla g$.

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \Delta g.$$

SO

$$\int_R \nabla f \cdot \nabla g \, dV + \int_R f \Delta g \, dV = \int_{\partial R} f \nabla g \cdot N \, d\sigma = \int_{\partial R} f \frac{\partial g}{\partial N} \, d\sigma.$$

SUBTRACTING:

$$\int_R f \Delta g - g \Delta f \, dV = \int_{\partial R} f \frac{\partial g}{\partial N} - g \frac{\partial f}{\partial N} \, d\sigma.$$

ALTERNATIVELY, CHOOSING $f = g$,

$$\int_R f \Delta f \, dV + \int_R |\nabla f|^2 \, dV = \int_{\partial R} f \frac{\partial f}{\partial N} \, d\sigma$$

IF f IS HARMONIC, $\Delta f = 0$,

$$\int_R |\nabla f|^2 \, dV = \frac{1}{2} \int_{\partial R} \frac{\partial}{\partial N} (f^2) \, d\sigma.$$