MAT 307 LECTURE 22:

FLUX INTEGRALS AND
THE DIVERGENCE THEOREM.

THE COORDINATE FUNCTIONS ARE $g_1(u,v)$, $g_2(u,v)$, $g_3(u,v)$.

THE <u>TANGENT SPACE</u> AT THE POINT $g(u,v)$ ON THE SURFACE IS SPANNED BY

g_u, g_v

WHICH WE ASSUME IS 2-DIMENSIONAL.

$g_u \times g_v$ IS CALLED THE <u>STANDARD NORMAL VECTOR</u> TO THE SURFACE.

THE <u>SURFACE AREA</u> IS

$$\sigma(S) = \iint_D |g_u \times g_v| \, du\, dv.$$

WE'RE FREQUENTLY ABLE TO REPRESENT A SURFACE AS A GRAPH OF A FUNCTION (IMPLICIT FUNCTION THEOREM)

$$\begin{pmatrix} u \\ v \end{pmatrix} \overset{g}{\longmapsto} \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}.$$

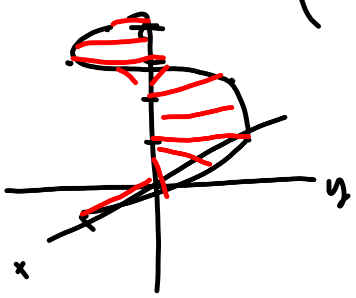
THEN $g_u = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}$, $g_v = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}$

$$g_u \times g_v = \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

$$|g_u \times g_v| = \sqrt{1 + f_u^2 + f_v^2}$$

EXAMPLE: A HELICOID SURFACE IS GIVEN BY

$$g(u,v) = \begin{pmatrix} u\cos v \\ u \sin v \\ v \end{pmatrix} \qquad 0 \le v \le 2\pi, \quad 0 \le u \le 1.$$



GIVE THIS MASS DENSITY $\mu(g(u,v)) = u$.

$$g_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \quad g_v = \begin{pmatrix} -u \sin v \\ u \cos v \\ 1 \end{pmatrix}.$$

$$g_u \times g_v = g_u \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + g_u \times \begin{pmatrix} -u\sin v \\ u \cos v \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin v \\ -\cos v \\ u \end{pmatrix}.$$

$$|g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}.$$

THE TOTAL MASS IS

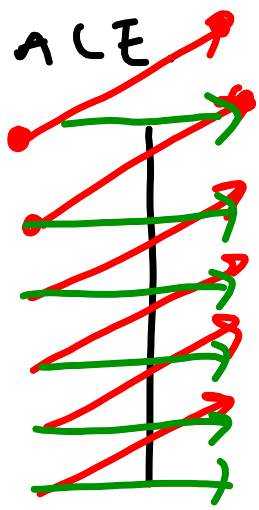
$$\int_{v=0}^{2\pi} \int_{u=0}^{1} \sqrt{1+u^2} \cdot u \, du \, dv \qquad w = 1+u^2$$

$$= 2\pi \cdot \frac{1}{2} \int_{w=1}^{2} \sqrt{w} \, dw$$

$$= \pi \cdot \frac{w^{3/2}}{3/2} \Big]_1^2$$

$$= \frac{2\pi}{3}\left(2^{3/2} - 1\right). \qquad \square$$

<u>FLUX</u> IS A MEASURE OF HOW MUCH A VECTOR FIELD FLOWS THROUGH A SURFACE



THE ORTHOGONAL TO SURFACE COMPONENT OF THE FIELD CONTRIBUTES TO RATE OF FLOW ACROSS THE SURFACE.

FLUX: $\int_S \vec{F} \cdot \underset{\sim}{N}\, d\sigma$, $\underline{N}$ THE NORMAL VECTOR

<u>EXAMPLE:</u> TRIVIAL EXAMPLE,
IF $F(u,v)$ IS TANGENT
TO THE SURFACE AT EVERY
POINT, THEN IT HAS 0
FLUX.

THE ORIENTATION

(COUNTERCLOCKWISE LOOPS)

IN PARAMETER SPACE

INDUCES AN ORIENTATION ON

THE PARAMETRIC SURFACE.

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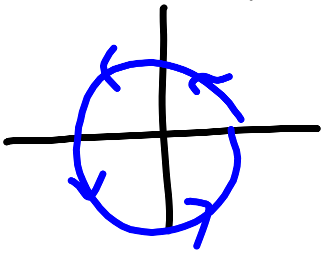
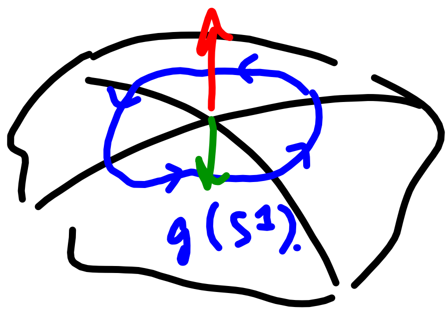




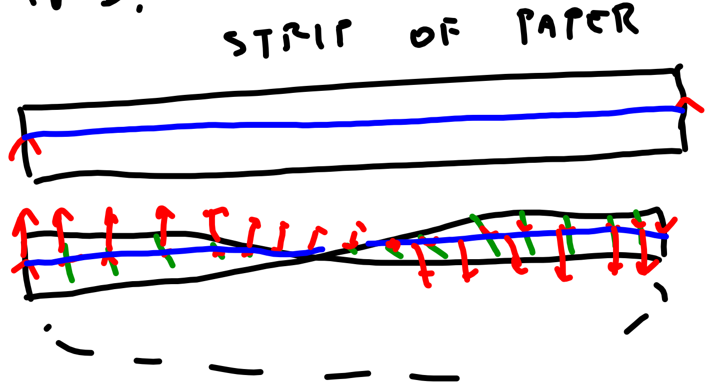
CHOOSE RED OR GREEN UNIT NORMAL

AN <u>ORIENTATION</u> ASSIGNS A DIRECTION FOR GOING AROUND LOOPS ON THE SURFACE.

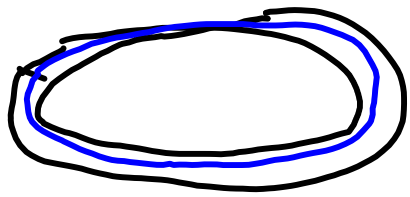
IT'S EQUIVALENT TO A CHOICE OF A UNIT NORMAL TO THE SURFACE.

NOT ALL SURFACES IN $\mathbb{R}^3$ ARE ORIENTABLE:

CLASSICAL EXAMPLE MÖBIUS BAND.



GLUE TWO ENDS TOGETHER.



TAKE A UNIT NORMAL AT EACH POINT OF THE PATH, VARYING CONTINUOUSLY

WHEN THE ENDS OF THE LOOP ARE GLUED TOGETHER, THE UNIT NORMALS POINT IN THE OPPOSITE DIRECTION.

THIS SHOWS THAT THE MÖBIUS BAND IS NOT ORIENTABLE.

GAUSS'S DIVERGENCE THEOREM:

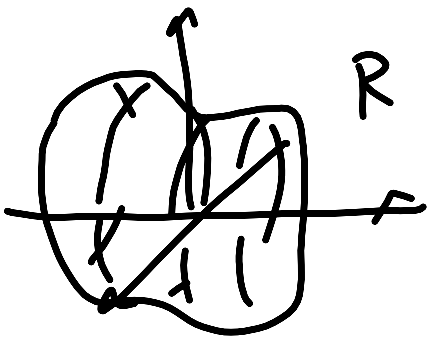
LET R BE A FINITE DISJOINT UNION OF SIMPLE CLOSED REGIONS HAVING POSITIVELY ORIENTED BOUNDARY, ∂R.

LET $\mathbb{F}$ BE A CONTINUOUSLY DIFFERENTIABLE VECTOR FIELD.

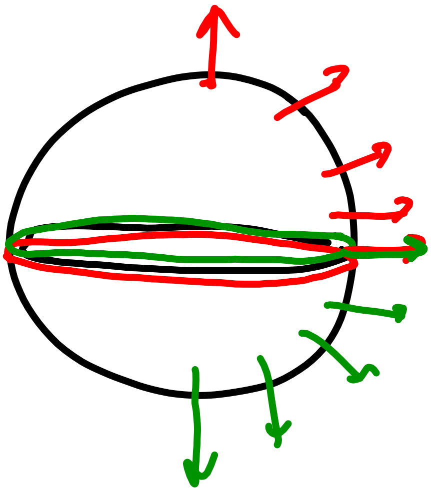
THEN $\int_R \operatorname{div} \mathbb{F} \, dV = \int_{\partial R} \underline{\mathbb{F}} \cdot dS$

FLUX.

$\int_{\partial R} \mathbb{F} \cdot N \, d\sigma$



WE MAY ASSUME THAT IN R,
y, z VARY IN A SET D, AND
THERE ARE TWO FUNCTIONS
$$g(y,z) \le x \le h(y,z)$$
DESCRIBING THE x COORDINATE.



THE UNIT NORMALS SHOULD MATCH IN THE OVERLAP REGION WHEN FOLLOWING ALONG A CURVE. $\square$

EXAMPLE: $\mathbb{F} = (0,0,1)$ CONSTANT

$\operatorname{div} \mathbb{F} = 0.$

THE FLUX OF $\mathbb{F}$ ACROSS THE PARABOLIC DISH

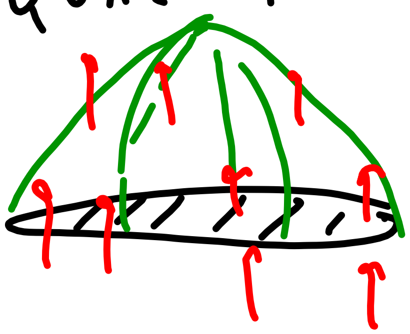
$\{(x, y, 1-x^2-y^2) : x^2+y^2 \le 1\}.$

IS EQUAL TO THE FLUX

ACROSS THE DISC

$\{(x,y,0) : x^2+y^2=1\}.$

EQUAL TO π.



FLUX OVER DISC:

$$\iint_{\text{Disc}} \mathbb{F} \cdot \underbrace{\mathbb{N}}_{\substack{\shortparallel \\ 1}} \, d\sigma$$

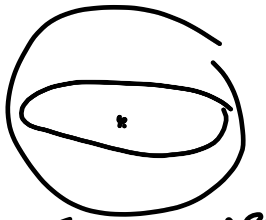
$=$ AREA OF DISC $= \pi$.

HARMONIC FUNCTIONS SATISFY THE MEAN VALUE PROPERTY

IF $S_{x,a} = \{ y : \|y - x\| = a \}$

SPHERE OF RADIUS a, CENTERED AT x, THEN,

$$\frac{1}{\text{AREA } S_{x,a}} \iint_{S_{x,a}} f(y)\, d\sigma = f(x).$$



THE VALUE OF f AT THE POINT IS EQUAL TO THE AVERAGE AROUND ANY SPHERE SURROUNDING THE POINT.

ONE WAY TO CHECK THIS IS TO SET $x = 0$, SAY, AND WRITE

$$f(t \cdot x) - f(s \cdot x) = \int \nabla f \cdot x$$

INTEGRATING OVER A SPHERICAL SHELL, THE INTEGRAL OF

$\underbrace{\nabla f \cdot x}_{\text{RADIAL DIRECTION,}}$ IS THE FLUX OF ∇f THROUGH THE SHELL, WHICH IS 0.

THIS CAN BE USED TO SHOW THAT THE MEAN VALUES OVER THE TWO SHELLS ARE EQUAL.

LET THE RADIUS OF THE SHELL TEND TO 0 AND USE THAT f IS CONTINUOUS