WE PROVED SEVERAL VERSIONS OF GREEN'S THEOREM LAST CLASS:

IF $\underline{F} = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ IS A VECTOR FIELD, AND γ IS A CLOSED LOOP ORIENTED COUNTER-CLOCKWISE, THEN

$$\int_\gamma \underline{F} \cdot d\underline{x} = \iint_R \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \, dA.$$

EXAMPLE: LET $\rho(\underline{x}, t)$

BE FLUID PRESSURE AT $\underline{x}$
(DENSITY)

AT TIME t, AND SUPPOSE

THE FLUID IS FLOWING

ACCORDING TO A VECTOR

FIELD $\mathbb{F}(x)$.

THE CONTINUITY EQUATION

$$\frac{\partial}{\partial t} \text{TOTAL MASS}$$

$$= \frac{\partial}{\partial t} \int_D \rho(\underline{x}, t) \, d\underline{x}$$

$$= - \int_{\partial D} \mathbb{F} \cdot \underline{N} \, ds$$

FLUX OUT OF REGION.

BY GREEN'S THEOREM

$$= - \int_D \operatorname{div} F \, dA$$

SO, EQUIVALENTLY

$$\int_D \frac{\partial \rho}{\partial t} - \operatorname{div} F \, dx \, dy = 0.$$

THIS IS TRUE FOR ALL REGIONS

D SO $\frac{\partial \rho}{\partial t} = \operatorname{div} F$.

REMARK: A POTENTIAL FUNCTION FOR A FIELD IS DETERMINED UP TO A CONSTANT, E.G. WE TALK ABOUT THE DIFFERENCE IN POTENTIAL BETWEEN TWO POINTS x_0, x, E.G. THE TWO POLES OF A BATTERY. THE PATH IS ESSENTIALLY THE SAME AS CONNECTING ONE POINT WITH A WIRE TO THE OTHER.

<u>RECALL</u>: IF $\mathbb{F} = \nabla f$ WE SAY $\mathbb{F}$ IS <u>CONSERVATIVE</u>.

THE FUNCTION f IS CALLED A <u>FIELD POTENTIAL</u>.

RECALL: $\mathbb{F} = m \cdot a(t)$

FORCE = MASS × ACCELERATION.

SUPPOSE $X(t)$ IS THE POSITION AT TIME t.

$$\int_{X(0)}^{X(1)} \mathbb{F} \cdot dx = \int_{t_0}^{t_1} m \cdot \underline{x}''(t) \cdot x'(t)\, dt$$

$$= \frac{m}{2} \int_{t_0}^{t_1} \frac{d}{dt} \|x'(t)\|^2 \, dt$$

$$= \frac{m}{2} \left(\|v(t_1)\|^2 - \|v(t_0)\|^2 \right).$$

THIS IS ALSO EQUAL TO THE CHANGE IN KINETIC ENERGY.

THIS PROVES: WHEN A PARTICLE MOVES SUBJECT TO A CONSERVATIVE FORCE FIELD

TOTAL ENERGY IS CONSERVED!

PROOF: ASSUME THE MASS OR PARTICLE IS LOCATED AT THE ORIGIN, SO THAT THE FIELD HAS FORM

$$c \cdot \frac{\underline{x}}{\|x\|^3}.$$

CONSIDER $f(\underline{x}) = \frac{1}{\|x\|}$

$$= \frac{1}{\sqrt{x_1^2 + \cdots + x_n^2}}.$$

$$\frac{\partial}{\partial x_i} \frac{1}{\sqrt{x_1^2 + \dots + x_n^2}} = \frac{-x_i}{(x_1^2 + \dots + x_n^2)^{3/2}}.$$

THUS $\nabla\left(\frac{1}{\|x\|}\right) = \frac{-x}{\|x\|^3}.$

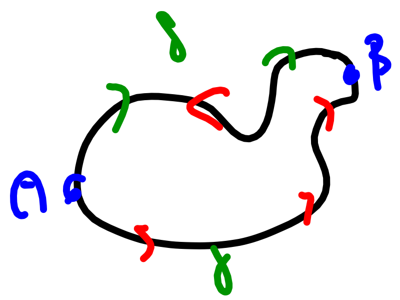
MULTIPLICATION BY A CONSTANT GIVES THE POTENTIAL FUNCTION.

PATH INDEPENDENCE:

WE SAY $\mathbb{F}$ IS PATH INDEPENDENT IF

$$\int_\gamma \mathbb{F}\, dx = \int_\delta \mathbb{F} \cdot dx$$

WHERE γ, δ HAVE THE SAME ENDPOINTS. PATH INDEPENDENCE IS EQUIVALENT TO $\int_\ell \mathbb{F} \cdot dx = 0$ FOR ANY CLOSED LOOP ℓ.



$$\int_\ell \mathbb{F} \cdot dx = \int_{-\delta} \mathbb{F} \cdot dx + \int_\gamma \mathbb{F} \cdot dx = 0.$$

THEOREM: IF $\mathbb{F}:\mathbb{R}^n \to \mathbb{R}^n$ IS A CONTINUOUSLY DIFF. GRADIENT FIELD THEN $\mathbb{F}'$ IS A SYMMETRIC MATRIX.

<u>THEOREM:</u> THE CONDITION curl $F = 0$ OR F' SYMMETRIC IS <u>NOT</u> SUFFICIENT TO GUARANTEE F IS CONSERVATIVE.

THEOREM: LET R BE AN OPEN COORDINATE RECTANGLE IN $\mathbb{R}^n$ AND $\mathbb{F}: R \to \mathbb{R}^n$ A CONTINUOUSLY DIFFERENTIABLE VECTOR FIELD WITH $\mathbb{F}'$ SYMMETRIC. THEN $\mathbb{F}$ IS CONSERVATIVE.

Ex: $\mathbb{F}(x,y) = (y^2, 2xy+1)$

$\text{curl}\,\mathbb{F} = 2y - 2y = 0.$

THUS $\mathbb{F}$ IS CONSERVATIVE.

TO CALCULATE A POTENTIAL FUNCTION

$$f(x,y) = \int \mathbb{F}(x,y)\,dx + \hat{f}(y).$$

$$\Rightarrow \frac{\partial}{\partial x} f(x,y) = \mathbb{F}(x,y) + \cancel{0},$$

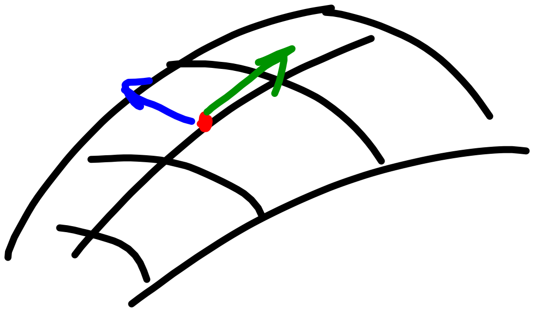
$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} \int \mathbb{F}(x,y)\,dx + \hat{f}'(y).$$

$$\int \mathbb{F}_1(x,y)\,dx = \int y^2\,dx = xy^2 + \tilde{f}(y)$$

$$\frac{\partial}{\partial y}\left(xy^2 + \tilde{f}(y)\right) = 2xy + 1$$

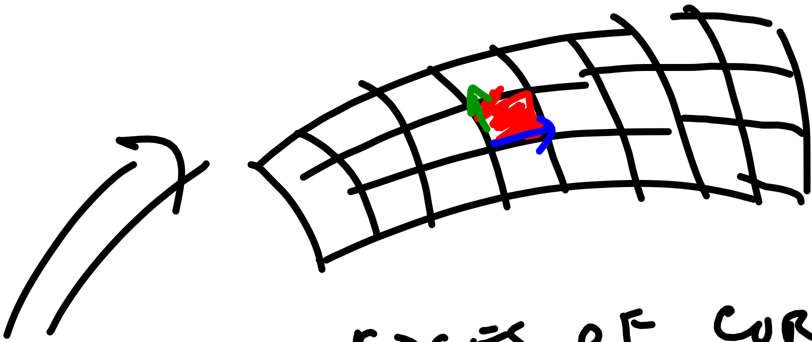
$$\Rightarrow \cancel{2xy} + \tilde{f}'(y) = \cancel{2xy} + 1$$

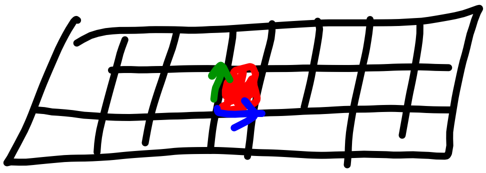
$\Rightarrow$ A VALID CHOICE IS $\tilde{f}(y) = y + c.$



g_u AND g_v ARE BOTH VECTORS TANGENT TO THE SURFACE AT THE POINT,

SO $g_u \times g_v$ IS PERPENDICULAR TO BOTH g_u, g_v, HENCE NORMAL TO THE SURFACE.





u,v PLANE
EACH BOX UNIT AREA

EDGES OF CURVY BOX $\simeq$ $g_u \times$ UNIT LENGTH
$g_v \times$ UNIT LENGTH.

$$\boxed{\text{RATIO OF AREAS} \simeq |g_u \times g_v|}$$

SURFACE AREA ELEMENT

$$dS = |g_u \times g_v| \, du \, dv.$$

<u>SURFACE AREA</u>: $\sigma(S) = \iint_{u,v} |g_u \times g_v| \, du \, dv.$

IF $\mu(x)$ IS A FN ON THE SURFACE,

$$\boxed{\int \mu \, dS = \iint_{u,v} \mu(g(u,v)) \, |g_u \times g_v| \, du \, dv}$$