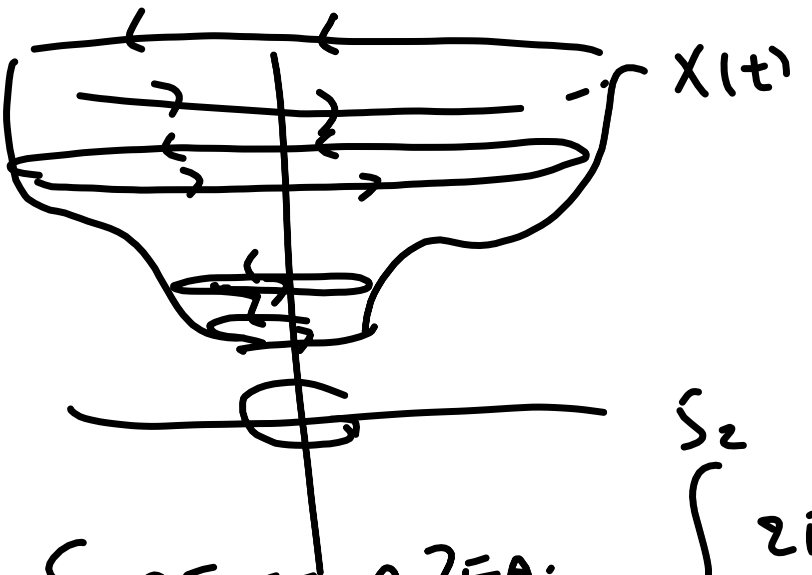
MAT 307 LECTURE 19

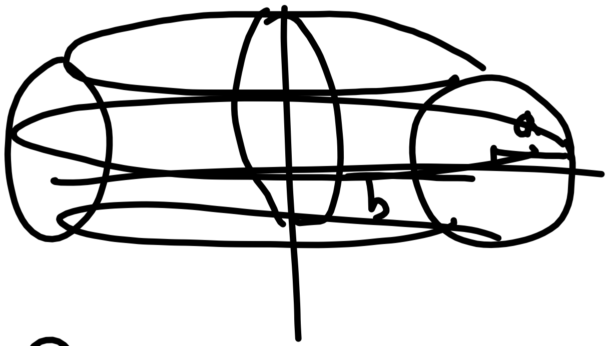
TANGENTIAL AND PRINCIPAL NORMAL VECTORS, CURVATURE



SURFACE AREA: $\displaystyle\int_{s_1}^{s_2} 2\pi\, r(s)\, ds$

r IS THE DISTANCE FROM THE AXIS OF ROTATION, S - ARC LENGTH PARAMETER.

EXAMPLE: SURFACE AREA OF A TORUS:



CIRCLE HAS COORDINATES

$(b+a\cos t, a\sin t)$

DISTANCE TO AXIS: $|b+a\cos t|$

ASSUME $b>a \Rightarrow = b+a\cos t$

$x'(t) = -a\sin t, \quad y'(t) = a\cos t$

$\sqrt{x'(t)^2+y'(t)^2} = a.$

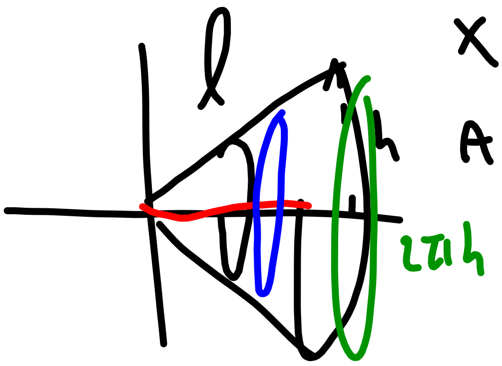
SURFACE AREA:

$$\sigma(S) = 2\pi \int_0^{2\pi} r(t)\sqrt{x'(t)^2+y'(t)^2}\,dt$$

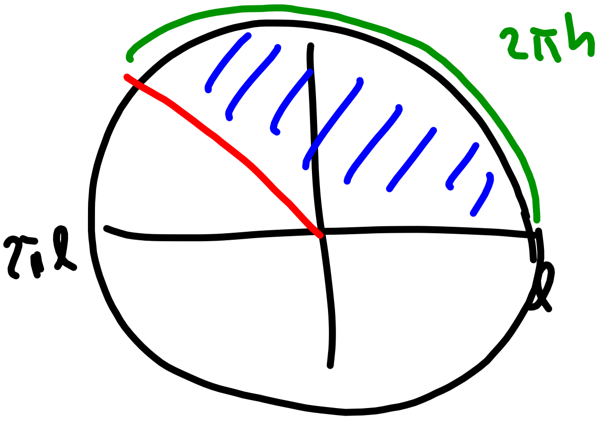
$$= 2\pi \int_0^{2\pi} (b+a\cos t)\cdot a\,dt.$$

$$\boxed{= 4\pi^2 ab.}$$

EXAMPLE: TAKE A LINE SEGMENT OF LENGTH ℓ FROM $(0,0)$ TO h UNITS ABOVE X-AXIS. ROTATE ABOUT X-AXIS TO FORM A CONE.



CUT A SLIT IN THE CONE AND UNWRAP IT



AREA:

$$\frac{2\pi h}{2\pi \ell}\cdot \pi \ell^2 = \pi h \ell.$$

WE HAVE

$$T(t)\cdot T(t) = \|T(t)\|^2 = 1.$$

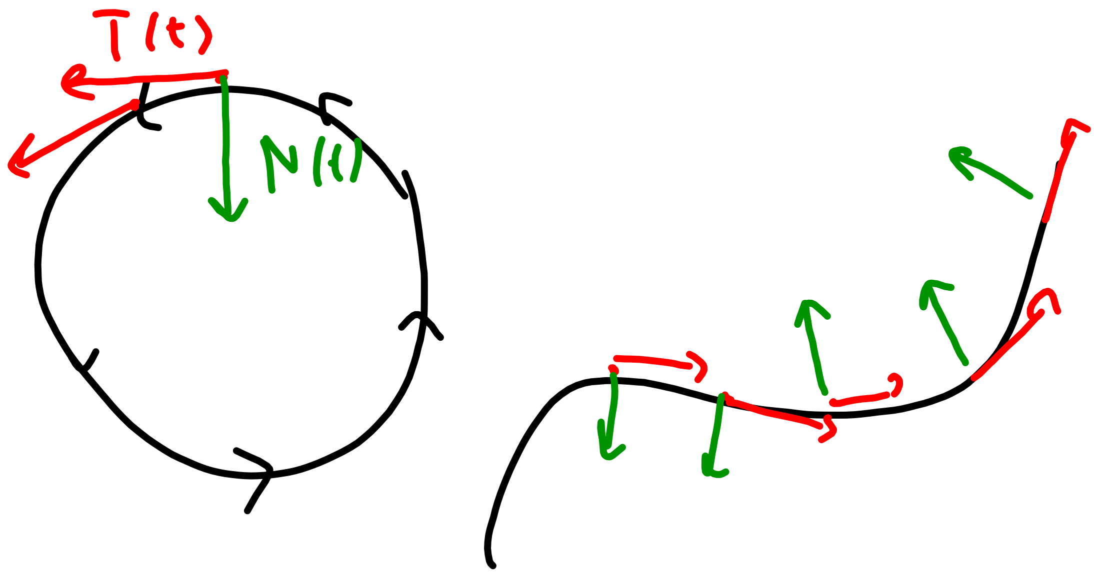
So $\frac{d}{dt}\left(T(t)\cdot T(t)\right) = 2T(t)\cdot T'(t)$

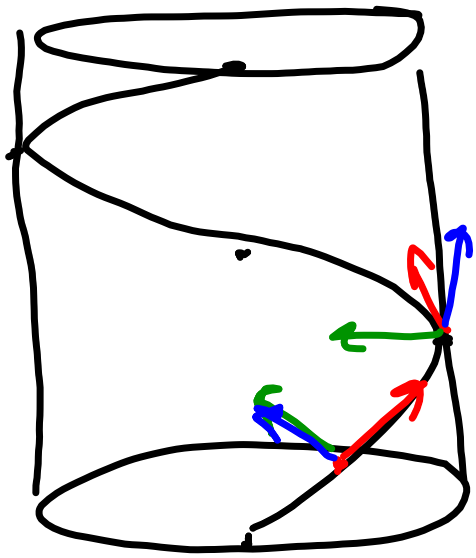
$= 0$

THUS $T'(t)$ IS $\perp$ TO $T(t)$.

IF $|T'(t)| \neq 0$ THEN DEFINE THE <u>PRINCIPAL NORMAL VECTOR</u>

$$N(t) = \frac{T'(t)}{|T'(t)|}.$$


T(t)
N(t)



$T(t)$

$N(t)$

$B(t)$

PROOF: WRITE

$$X'(t) = \frac{ds}{dt} \cdot T(t).$$

Thus,

$$X''(t) = \frac{d^2 s}{dt^2} T(t) + \frac{ds}{dt} T'(t)$$

$$= s'' T(t) + s' |T'(t)| N(t). \quad \square$$

THE SCALAR CURVATURE IS A MEASURE OF HOW QUICKLY THE CURVE IS TURNING AT THE POINT.

<u>EXAMPLE</u>: CIRCLE OF RADIUS R, PARAM BY $x(s) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}\right)$.

$$x'(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right)$$

$|x'(s)| = 1 \Rightarrow$ ARC LENGTH PARAMETERIZATION

$$T(s) = x'(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right)$$

$$\frac{dT}{ds} = \frac{1}{R}\left(-\cos\frac{s}{R}, -\sin\frac{s}{R}\right)$$

$$N(s) = \left(-\cos\frac{s}{R}, -\sin\frac{s}{R}\right)$$

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{1}{R}.$$

<u>PROOF:</u> BY THE CHAIN RULE

$$\frac{dT}{ds} = \frac{dT}{dt} \cdot \frac{dt}{ds}$$

SO $K = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\right| \cdot 1 / \left|\frac{ds}{dt}\right|$

OR $\left|\frac{dT}{dt}\right| = K \cdot |s'| = Ks'$.

SINCE $s' > 0$,

$$X'' = s'' T(t) + s' \left|\frac{dT}{dt}\right| \cdot N(t)$$

$$= s'' T(t) + s' \cdot K \cdot s' \cdot N(t)$$

$$= s'' T(t) + K(s')^2 N(t).$$

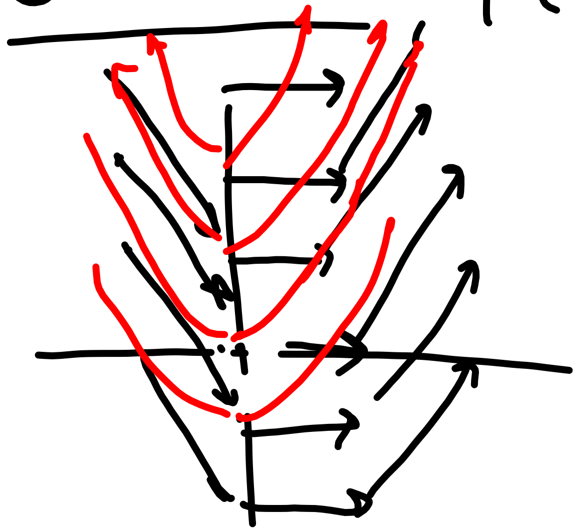
$\square$

DEFINITION: A FLOW LINE IS A SOLUTION OF THE DIFFERENTIAL EQUATION

$$x'(t) = F(x(t)).$$

WHERE $X : [a,b] \to \mathbb{R}^n$ IS A SMOOTH CURVE AND $F : \mathbb{R}^n \to \mathbb{R}^n$ IS A VECTOR FIELD.

EXAMPLE: $F(x,y) = \begin{pmatrix} 1 \\ 2x \end{pmatrix}$



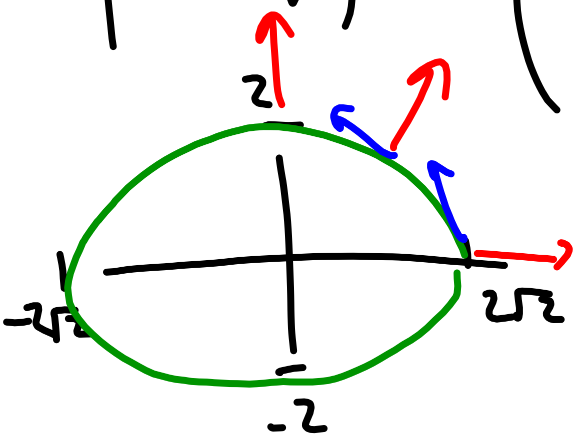
$x(t) = \begin{pmatrix} t \\ t^2+c \end{pmatrix}$

$x'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$

SATISFIES THIS

Example: $f(x,y) = \frac{1}{8}x^2 + \frac{1}{4}y^2$

$$F = \nabla f = \begin{pmatrix} \frac{x}{4} \\ \frac{y}{2} \end{pmatrix}.$$



$f(\underline{x}) = 1.$

$\rightarrow \nabla f.$

EXAMPLE:

$$F(x,y) = \begin{pmatrix} xy \\ x - y^2 \end{pmatrix}$$

$$\text{Div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = y - 2y = -y$$

EXAMPLE:

$$F(x,y) = \begin{pmatrix} \frac{1}{8}(-x+y) \\ \frac{1}{8}(-x-y) \end{pmatrix}$$

$$\operatorname{div} F = -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}.$$

A SECOND OPERATION ON VECTOR FIELDS IS THE CURL

$F: \mathbb{R}^3 \to \mathbb{R}^3$, $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$

$$\text{Curl } F = \nabla \times F$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \underline{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \underline{k}.$$

EXAMPLE:

$$\underline{F}(x,y) = \begin{pmatrix} \frac{1}{8}(-x+y) \\ \frac{1}{8}(-x-y) \end{pmatrix}$$

$$\text{Curl } F = \left(-\frac{1}{8}\right) - \frac{1}{8} = -\frac{1}{4}.$$