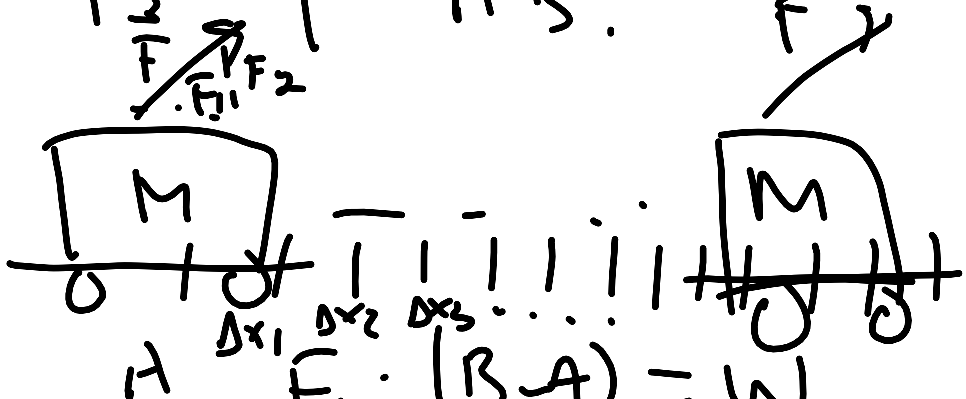
MAT 307 LECTURE 18

LINE INTEGRALS.

SURFACE AREA

MOTIVATION: IN CLASSICAL MECHANICS, THE WORK DONE BY A CONSTANT FORCE $\vec{F}$ IN MOVING AN OBJECT FROM A TO B IS $\vec{F} \cdot \overrightarrow{AB}$.



$\vec{F}_1 \cdot (B-A) = W$.

$$W \approx \sum_i \vec{F}_i(x_i) \cdot \Delta x_i$$

"LINE INTEGRAL"

THIS ALLOWS FOR A CURVED PATH IN WHICH THE Δx_i HAVE DIFFERENT DIRECTION.

DEFINITION: LET $F: \mathbb{R}^n \to \mathbb{R}^n$ A VECTOR FIELD, WHICH IS CONTINUOUS, LET $g: [a,b] \to \mathbb{R}^n$ BE A PARAMETRIC CURVE WITH CONTINUOUS DERIVATIVE, PARAMETERIZING γ. THE LINE INTEGRAL OF F OVER γ IS

$$\int_\gamma F = \int_a^b F(g(t)) \cdot g'(t)\,dt.$$

REMARK: WE'LL SHOW THAT THIS IS INDEPENDENT OF THE CHOICE OF PARAMETERIZATION SO LONG AS THE PARAMETERIZATION IS SMOOTH.

<u>WORK:</u> THE WORK DONE BY A FORCE FIELD $\vec{F}$, AS AN OBJECT FOLLOWS PATH $\underline{x}(t)$, $t_0 \leq t \leq t_1$ IS

$$W = \int_{t_0}^{t_1} F(x(t)) \cdot x'(t)\,dt$$

$$= \int_{\gamma} \vec{F}(x) \cdot d\underline{x}.$$

<u>EXAMPLE:</u> $F(x,y,z) = \begin{pmatrix} -y \\ x \\ 1 \end{pmatrix}$

$$x(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

$$F(x(t)) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

$$x'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

$$F(x(t)) \cdot x'(t) = \sin^2 t + \cos^2 t + 1 = 2$$

$$\int_a^b F(x(t)) \cdot x'(t)\, dt = \int_a^b 2\, dt = 2(b-a).$$

THEOREM: EQUIVALENT PARAM. OF CURVE GIVE EQUAL LINE INTEGRALS.

PROOF:
$$\int_\gamma F\,dx = \int_c^d F(g(u))\,g'(u)\,du$$

$$g(u) = g(u(t)), \quad u = u(t)$$

$$\frac{d}{dt}\, g(u(t)) = g'(u(t))\cdot u'(t).$$

BY THE CHANGE OF VARIABLES FORMULA FOR REAL VALUED INTEGRALS,

$$\int_c^d \underbrace{F(g(u))\cdot g'(u)}_{w(u)}\,du = \int_a^b w(u(t))\,u'(t)\,dt$$

$$= \int_a^b F(g(u(t)))\cdot g'(u(t))\,u'(t)\,dt$$

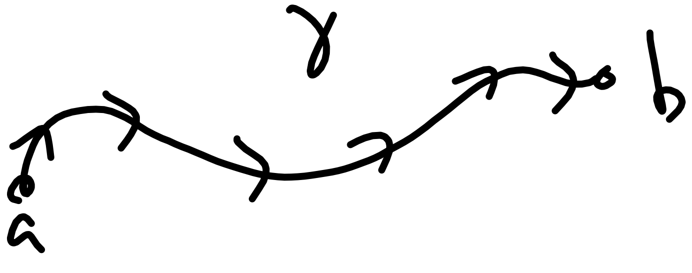
$$= \int_a^b F(f(t))\cdot f'(t)\,dt.$$

$\square$

<u>THEOREM:</u> SUPPOSE $F = \nabla f$ IS A CONTINUOUS CONSERVATIVE VECTOR FIELD, AND γ IS A SMOOTH CURVE. THEN

$$\int_\gamma \nabla f \cdot dx = f(b) - f(a)$$

WHERE b IS THE <u>TERMINAL ENDPOINT</u>, a IS THE <u>INITIAL ENDPOINT</u>.



EXAMPLE: SUPPOSE

$f(x,y) = \frac{1}{2}(x^2+y^2)$

$\nabla f(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}.$

THEN $\displaystyle\int_\gamma \nabla f \cdot dx$

$= f(\gamma(1)) - f(\gamma(0))$

$= \displaystyle\int_{(x_1,y_1)}^{(x_2,y_2)} x\,dx + y\,dy$

$= \dfrac{x_2^2+y_2^2}{2} - \left(\dfrac{x_1^2+y_1^2}{2}\right).$

<u>THEOREM</u>: IF $\vec{F}$ IS A <u>CONSERVATIVE</u> VECTOR FIELD, THE LINE INTEGRAL OF $\vec{F}$ ABOUT ANY CLOSED LOOP IS 0.

<u>PROOF</u>: THIS FOLLOWS BECAUSE THE INITIAL AND FINAL ENDPOINTS ARE EQUAL.

CONSTANT SPEED PARAMETERIZATION

$$S(t) = \int_{t_0}^{t} |x'(u)|\, du$$

"DISTANCE TRAVELED TO TIME t"

WHEN THE CURVE IS PARAM BY S WE SAY IT HAS THE "ARC LENGTH" PARAMETERIZATION

$S(t)$ CAN BE INVERTED TO OBTAIN $T(s)$.

$$\frac{d}{ds}\underline{X}(s) = \frac{d\underline{x}}{dt} \cdot \frac{dt}{ds}$$

VECTOR SCALAR

$$\left|\frac{d}{ds}\underline{X}(s)\right| = \left|\frac{d\underline{x}}{dt}\right| \cdot \frac{dt}{ds}$$

$$\left|\frac{d\underline{x}}{dt}\right| = \frac{ds}{dt}$$

So $\frac{ds}{dt} \cdot \frac{dt}{ds} = 1.$

THUS THE ARC LENGTH PARAMETERIZATION GIVES A CONSTANT SPEED PARAM. OF γ.

EXAMPLE:

GIVEN $g(t) = \left(t, \frac{2}{3} t^{3/2}\right)$.

$$g'(t) = \left(1, t^{1/2}\right).$$

$$S(t) = \int_0^t |g'(u)|\, du$$

$$= \int_0^t \sqrt{1+u}\, du$$

$$= \frac{2}{3}(1+u)^{\frac{3}{2}} \Big]_0^t$$

$$= \frac{2}{3}\left((1+t)^{\frac{3}{2}} - 1\right).$$

$$s = \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

$$\frac{3}{2}s + 1 = (1+t)^{3/2}$$

OR $t = \left(\frac{3}{2}s + 1\right)^{\frac{2}{3}} - 1.$

THUS, THE ARC-LENGTH PARAMETERIZATION IS

$$h(s) = g(t(s))$$

$$= \left(t, \frac{2}{3} t^{3/2}\right)$$

$$= \left(\left(\frac{3}{2}s + 1\right)^{2/3} - 1, \frac{2}{3}\left[\left(\frac{3}{2}s + 1\right)^{\frac{2}{3}} - 1\right]^{3/2}\right).$$

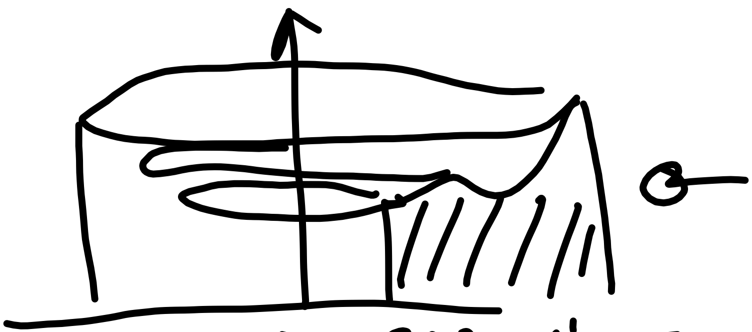
TO OBTAIN THE MASS WITH A GENERAL PARAMETERIZATION $g(t)$, USE $\frac{ds}{dt} = |g'(t)|$.

THIS GIVES, WITH $h(s) = g(\tau)$,

$$\int_{t_0}^{t_1} \mu(g(t)) \cdot \frac{ds}{dt}\, dt$$

$$= \int_{t_0}^{t_1} \mu(g(t)) \cdot |g'(t)|\, dt.$$

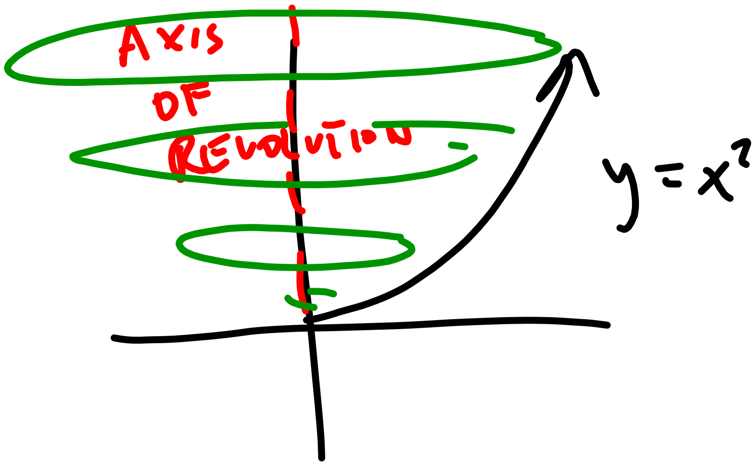
SURFACES OF REVOLUTION:



MASS
OF REVOLUTION

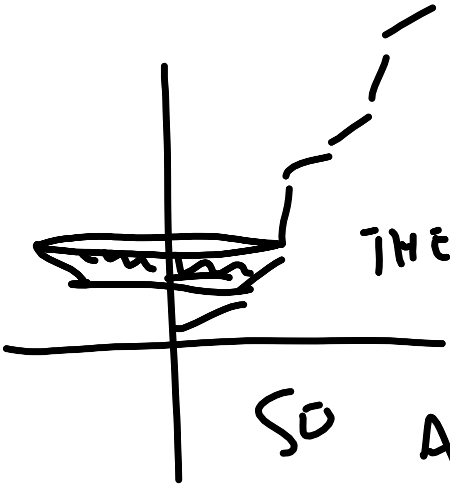
VOLUME - PAPPUS' THM.

THE SURFACE AREA IS THE AREA OF A SURFACE OF REVOLUTION.



DURING REVOLUTION, A POINT AT DISTANCE $R(s)$ FROM THE LINE

TRACES OUT A CIRCLE OF RADIUS $2\pi R(s)$. IN CIRCUMFERENCE.

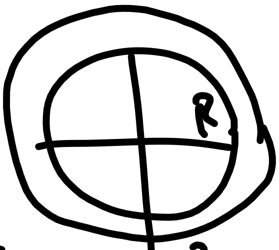


THE CHANGE IN RADIUS IS NEGLIGIBLE

SO AREA OF ONE SEGMENT IS $\approx R(s) \cdot \Delta(s) \cdot 2\pi$.

IT IS EXACTLY THIS IF THE SEGMENT IS VERTICAL. IF HORIZONTAL,

AREA:



$\pi(R+\Delta S)^2 - \pi R^2 = 2\pi R \Delta S + \cancel{\pi(\Delta S)^2}$

→ 0 IN ΔS → 0 LIMIT.