MAT 307 LECTURE 18

PROOF:

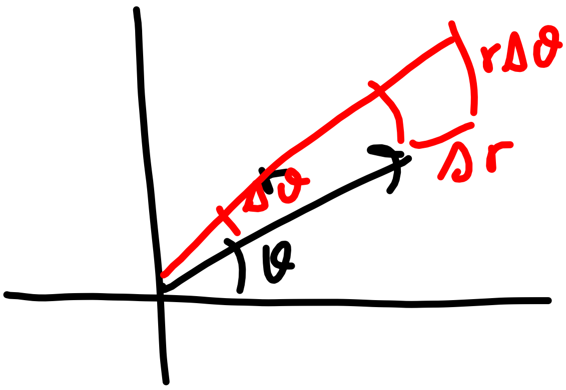
$$\int_a^b \int_c^y \frac{\partial}{\partial y} g(x,y)\, dy\, dx$$

$$= \int_a^b g(x,y) - g(x,c)\ dx$$

TAKING THE DERIVATIVE OUTSIDE WITH RESPECT TO y OBTAINS THE RESULT.

CHANGE OF VARIABLES:

POLAR COORDINATES:



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

AREA ELEMENT

$r\,\Delta r\,\Delta\theta$

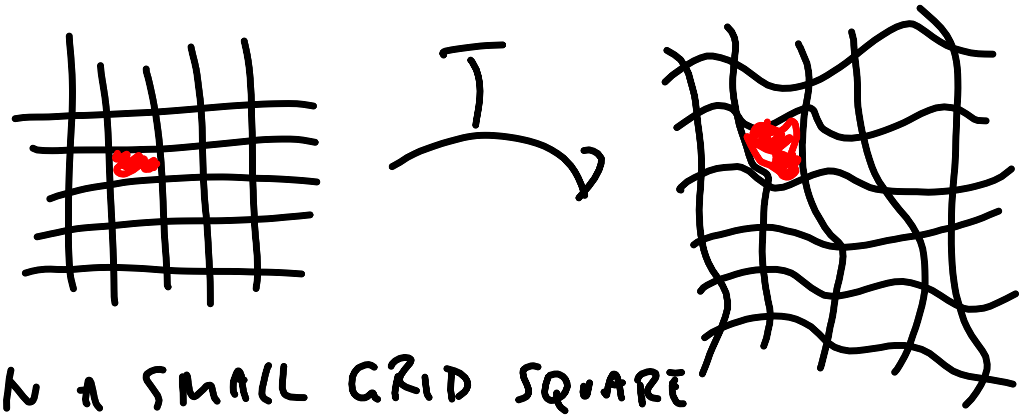
$$\int_D f(x,y)\,dx\,dy = \int_E f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

CYLINDRICAL COORDINATES:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

AREA ELEMENT: $r\,dr\,d\theta\,dz$.

SKETCH OF THE IDEA:



ON A SMALL GRID SQUARE

T IS WELL APPROXIMATED BY THE DERIVATIVE T', WHICH IS LINEAR

$$\text{AREA}(T(\text{SQUARE})) \simeq |\det T'| \ \text{AREA SQUARE}.$$

POLAR COORDINATES:

$$T: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$T' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\boxed{\begin{aligned} \det T' &= r\cos^2\theta + r\sin^2\theta \\ &= r \end{aligned}}$$

SPHERICAL COORDINATES:

$$T: \begin{pmatrix} \rho \sin\varphi \cos\theta \\ \rho \sin\varphi \sin\theta \\ \rho \cos\varphi \end{pmatrix}$$

$$T': \begin{pmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix}$$

$$\det T' = \rho^2 \left\{ \sin\varphi\sin\theta \begin{vmatrix} \sin\varphi\sin\theta & \cos\varphi\sin\theta \\ \cos\varphi & -\sin\varphi \end{vmatrix} \right.$$

$$\left. + \sin\varphi\cos\theta \begin{vmatrix} \sin\varphi\cos\theta & \cos\varphi\cos\theta \\ \cos\varphi & -\sin\varphi \end{vmatrix} \right\}$$

$$= \rho^2 \left\{ \sin\varphi \sin^2\theta \, \big(\underbrace{-\sin^2\varphi - \cos^2\varphi}_{-1}\big) + \sin\varphi \underbrace{\cos^2\theta}_{1} \big(\underbrace{-\sin^2\varphi - \cos^2\varphi}_{-1}\big) \right\}$$

$$= -\rho^2 \sin\varphi.$$

$$|\det T'| = \rho^2 \sin\varphi.$$

CENTROIDS AND MOMENTS:

$\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$, POSITIONS.

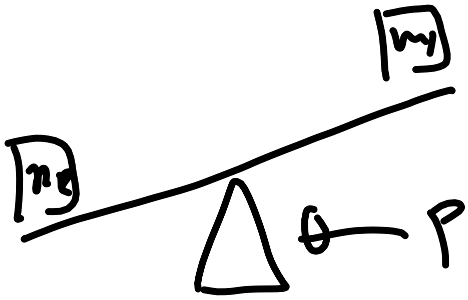
$m_1, m_2, \ldots, m_N$ MASSES.

TOTAL MASS: $M = m_1 + \ldots + m_N$.

CENTER OF MASS IS THE WEIGHTED AVERAGE OF POSITIONS

$$\bar{\underline{x}} = \frac{1}{M}\left(m_1 \underline{x}_1 + m_2 \underline{x}_2 + \ldots + m_N \underline{x}_N\right).$$

THIS IS THE LOCATION IN SPACE ON WHICH THE MASSES PLACED ON A WEIGHTLESS PLATE WOULD BALANCE.



PIVOT.

THEOREM: LET P AN ARBITRARY PLANE CONTAINING CENTER OF MASS $\bar{X}$, THEN $M_P = 0$.

FOR A MASS DENSITY

$$\mu(\underline{x}) \geq 0$$

THE <u>CENTER OF MASS</u> IS

$$\underline{\bar{x}} = \frac{1}{M(B)} \cdot \int_B \mu(\underline{x}) \cdot \underline{x} \, d\underline{x}.$$

$$M(B) = \int_B \mu(\underline{x}) \, d\underline{x}.$$

<u>E.G. IN 3 DIMENSIONS:</u>

$$\bar{x} = \frac{1}{M(B)} \int_B x \, \mu(x,y,z) \, dV$$

$$\bar{y} = \frac{1}{M(B)} \int_B y \, \mu(x,y,z) \, dV$$

$$\bar{z} = \frac{1}{M(B)} \int_B z \, \mu(x,y,z) \, dV.$$

TO FIND THE AVERAGE

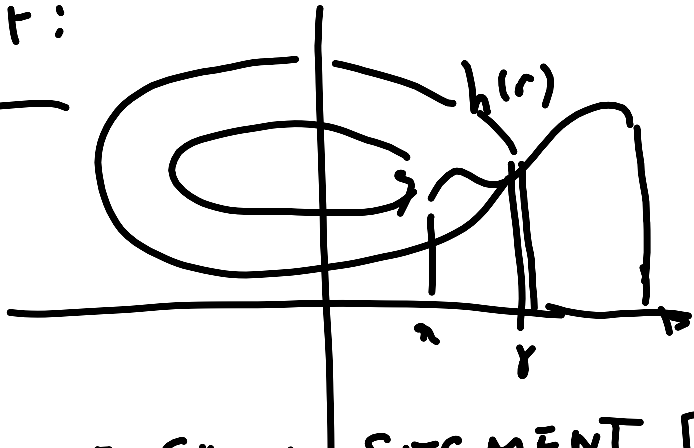
X - COORDINATE

$$\int_Q \underbrace{x}_{r\cos\theta} \underbrace{\mu(x,y)}_{r^2}\, dx\, dy = \int_0^a \int_0^{\pi/2} r^3 \cos\theta \cdot r \, d\theta\, dr$$

$$= 1 \cdot \int_0^a r^4\, dr = \frac{r^5}{5}\Big]_0^a = \frac{a^5}{5}.$$

$$\bar{X} = a^5/5 \Big/ \frac{\pi a^4}{8} = \boxed{\frac{8a}{5\pi}}$$

PROOF:



VOLUME OF SMALL SEGMENT ROTATED

IS $2\pi r \cdot h(r)\Delta r$.

VOLUME: $\int_a^b 2\pi r\, h(r)\, dr$

MEANWHILE: THE AVERAGE

x COORDINATE OF THE BODY

IS
$$\frac{\int_a^b r \cdot h(r)\, dr}{\int_a^b h(r)\, dr} = \bar{r}$$

AREA OF REGION

$$\boxed{\text{VOLUME: } 2\pi\, \bar{r} \cdot \text{AREA OF REGION.}}$$

$$\int_{D_\delta} \frac{1}{\sqrt{x^2+y^2}}\, dx = \int_0^{2\pi}\int_\delta^{2} \frac{1}{r}\, r\, dr\, d\theta$$

$$= 2\pi \cdot \int_\delta^{1} dr = 2\pi(1-\delta).$$

$$\lim_{\delta \downarrow 0} 2\pi(1-\delta) = \boxed{2\pi}.$$

EXAMPLE:

$$\int_0^\infty e^{-x}\,dx = \lim_{N\to\infty} \int_0^N e^{-x}\,dx$$

$$= \lim_{N\to\infty} 1 - e^{-N} = \underline{1}.$$

NORMAL DISTRIBUTION:

$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ STANDARD NORMAL.

$$\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)^2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}} dx\, dy.$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} r e^{-\frac{r^2}{2}} dr\, d\theta.$$

$u = r^2/2$
$du = r\, dr.$

$$= \boxed{\int_0^{\infty} e^{-u} du = 1.}$$

THE NORMAL OF MEAN μ, VARIANCE σ^2 IS

$$\boxed{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

σ - STANDARD DEVIATION

μ - MEAN, CENTER, TRANSLATED DISTRIBUTION.