RECALL FROM LAST LECTURE:

THEOREM: LET $f: \mathbb{R}^n \to \mathbb{R}$ TWICE CTSLY DIFF. WITH CRITICAL POINT AT $\underline{x}_0$.

(1) IF $\frac{\partial^2}{\partial u^2} f(x_0) > 0$ ALL UNIT VECTORS u, THEN x_0 IS A LOCAL MIN

(2) IF $\frac{\partial^2}{\partial u^2} f(x_0) < 0$ ALL UNIT VECTORS u, THEN x_0 IS A LOCAL MAX

(3) IF THERE ARE UNIT VECTORS $\underline{u}$ FOR WHICH $\frac{\partial^2}{\partial u^2} f(\underline{x}_0)$ TAKES POS. AND NEG. VALUES, THEN SADDLE POINT.

PROOF: THE HESSIAN MATRIX

IS $H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$.

So $\frac{\partial^2 f}{\partial \underline{u}^2} = \underline{u}^t H \underline{u}$

$$= \frac{\partial^2 f}{\partial x^2} u_1^2 + 2 \frac{\partial^2 f}{\partial x \partial y} u_1 u_2 + \frac{\partial^2 f}{\partial y^2} u_2^2 .$$

$\square$

PROOF: GIVEN THE QUADRATIC FORM

$$f_{xx} u_1^2 + 2 f_{xy} u_1 u_2 + f_{yy} u_2^2$$

COMPLETE THE SQUARE TO WRITE THIS AS (ASSUMING $f_{xx} \neq 0$)

$$f_{xx}\left(u_1 + \frac{f_{xy}}{f_{xx}} u_2\right)^2 + \left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right) u_2^2$$

OR $f_{xx}\left(f_{xx} u_1^2 + 2 f_{xy} u_1 u_2 + f_{yy} u_2^2\right)$

$$= \left(f_{xx} u_1 + f_{xy} u_2\right)^2 + D u_2^2$$

IF $D > 0$, THEN THE VALUES ARE ALL NON-NEGATIVE.

IN THIS CASE, IF $f_{xx} > 0$ THEN $\frac{\partial^2 f}{\partial u^2} > 0$ ALL UNIT VECTORS $\underline{u}$

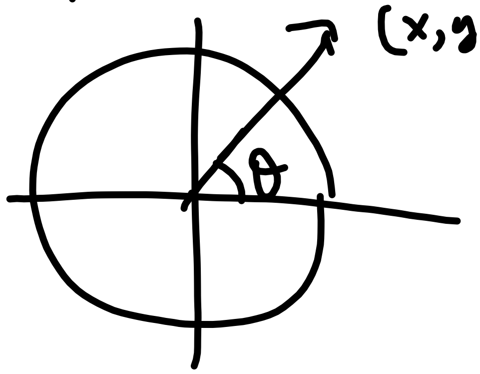
SO LOCAL MIN.

IF $f_{xx} < 0$, THEN $\frac{\partial^2 f}{\partial u^2} < 0$ ALL UNIT VECTORS $\underline{u}$, SO LOCAL MAX.

IF $D < 0$ THEN POSITIVE AND NEGATIVE VALUES GET TAKEN $\rightarrow$ SADDLE POINT.

OTHER COORDINATE SYSTEMS.

POLAR COORDINATES.



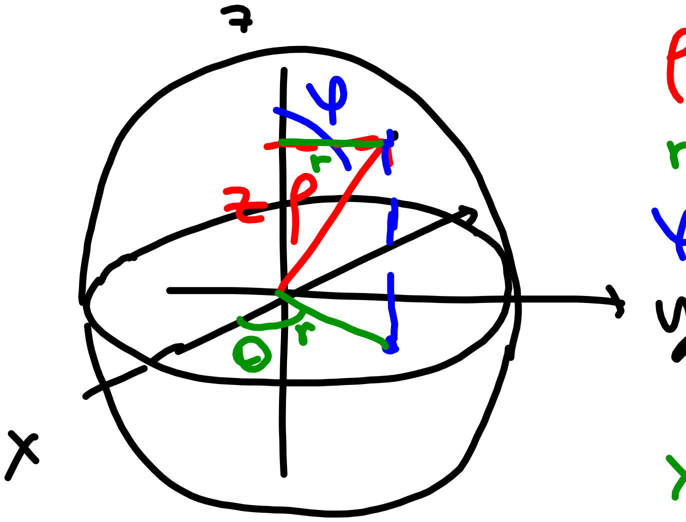
$x = r\cos\theta$

$y = r\sin\theta$

$\theta = \tan^{-1}(y/x)$, ANGLE WITH x AXIS.

$r = \sqrt{x^2+y^2}$.

SPHERICAL COORDINATES:



$\rho = \sqrt{x^2+y^2+z^2}$

$r = \sqrt{x^2+y^2}$

φ = ANGLE FROM NORTH POLE

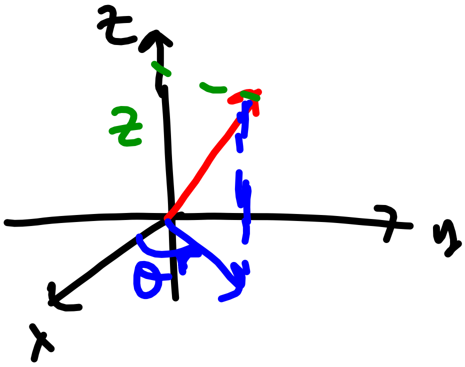
θ = ANGLE IN xy-PLANE AS IN POLAR COORDINATES.

$z = \rho \sin\varphi$ $\quad r = \rho\cos\varphi$

"LONGITUDE AND LATTITUDE"
θ $\qquad \varphi$

CYLINDRICAL COORDINATES:

THIS TRACKS THE PROJECTION TO THE xy-PLANE IN POLAR COORDINATES (r,θ), ALONG WITH THE z COORDINATE.



$$\begin{pmatrix} r \\ \theta \\ z \end{pmatrix} \longleftrightarrow \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}.$$

EXAMPLE: GIVEN THE POINT $(\cos\theta, \sin\theta)$ ON THE UNIT CIRCLE WITH $r=1$, THE DERIVATIVE WITH RESPECT TO θ IS

$$(-\sin\theta, \cos\theta)^t$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\frac{d}{d\theta}\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ ← DERIVATIVE IN POLAR COORDINATES.

GIVEN A FUNCTION $f(x,y)$ OF TWO VARIABLES, HOLD x FIXED AND INTEGRATE IN y TO OBTAIN

$$F(x) = \int_c^d f(x,y)\,dy.$$

THE <u>ITERATED INTEGRAL</u> IS

$$\int_a^b F(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx.$$

WE CAN ALSO INTEGRATE IN THE OTHER ORDER,

$$\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy.$$

WE WANT TO UNDERSTAND THIS AS AN AREA INTEGRAL OF $f(x,y)$ OVER $[a,b]\times[c,d]$.

ITERATED INTEGRALS MAY ALSO BE PERFORMED OVER NON-RECTANGULAR REGIONS:

$$\int_0^1 \left[\int_0^{1-x^2} (x+y)\,dy \right] dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{1-x^2} dx$$

$$= \int_0^1 \left(x(1-x^2) + \frac{(1-x^2)^2}{2} \right) dx$$

$$= \int_0^1 x - x^3 + \frac{1}{2} - x^2 + \frac{x^4}{2}\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10} \right]_0^1$$

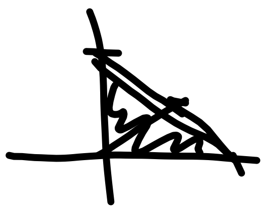
$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{31}{60}.$$

EXAMPLE:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx$$

$x+y+z \leq 1$

$x, y, z > 0$



$$= \int_0^1 \int_0^{1-x} x(1-x-y) \, dy \, dx$$

$$= \int_0^1 \left[xy - x^2y - \frac{xy^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 x(1-x) - x^2(1-x) - \frac{x(1-x)^2}{2} \, dx$$

$$= \frac{1}{24}.$$

DEFINITION: A COORDINATE RECTANGLE IS A SET OF FORM

$$\{ \underline{x} = (x_1, x_2, \dots, x_n),\ a_i \le x_i \le b_i \}.$$

$i = 1, 2, \dots, n.$

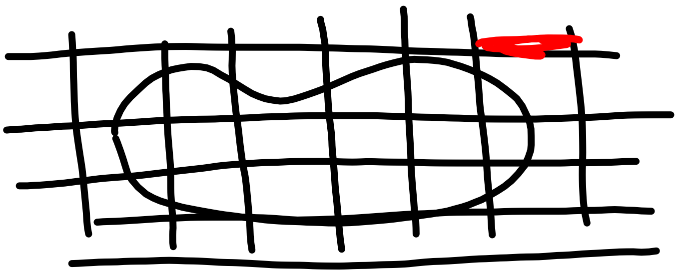
WE SAY R IS <u>DEGENERATE</u>
IF $\mathrm{VOL}(R) = 0$.

$B \subset \mathbb{R}^n$ IS <u>BOUNDED</u> IF
THERE EXISTS SOME REAL
NUMBER N SO THAT,
FOR ALL $\underline{x} \in B$, $\|\underline{x}\| \leq N$.

A FINITE COLLECTION OF $n-1$ DIMENSIONAL HYPERPLANES IS CALLED A <u>GRID</u> OF $\mathbb{R}^n$. THESE CUT $\mathbb{R}^n$ INTO FINITELY MANY BOUNDED RECTANGLES AND SOME UNBOUNDED REGIONS.

A GRID <u>COVERS</u> A SET $S \subset \mathbb{R}^n$ IF ALL OF S IS CONTAINED IN THE BOUNDED RECTANGLES.

THE MAXIMUM LENGTH OF THE RECTANGLES COVERING S IS CALLED THE MESH.



MAXIMUM LENGTH

<u>DEFINITION:</u> WE SAY f IS <u>RIEMANN INTEGRABLE</u> OVER S IF

$$\lim_{\text{MESH} \to 0} \sum_{R_i} f(x_i)\,\text{VOL}(R_i)$$

CONVERGES TO A LIMIT, WHICH IS THEN CALLED

$$\boxed{\int_S f \, dV.}$$

THEOREM: LET $f: \mathbb{R}^n \to \mathbb{R}$

DEFINED AND BOUNDED ON A SET

S, SUCH THAT

(i) ∂S HAS ZERO CONTENT

(ii) f IS CONTINUOUS EXCEPT POSSIBLY ON A SET OF ZERO CONTENT.

THEN f IS INTEGRABLE.