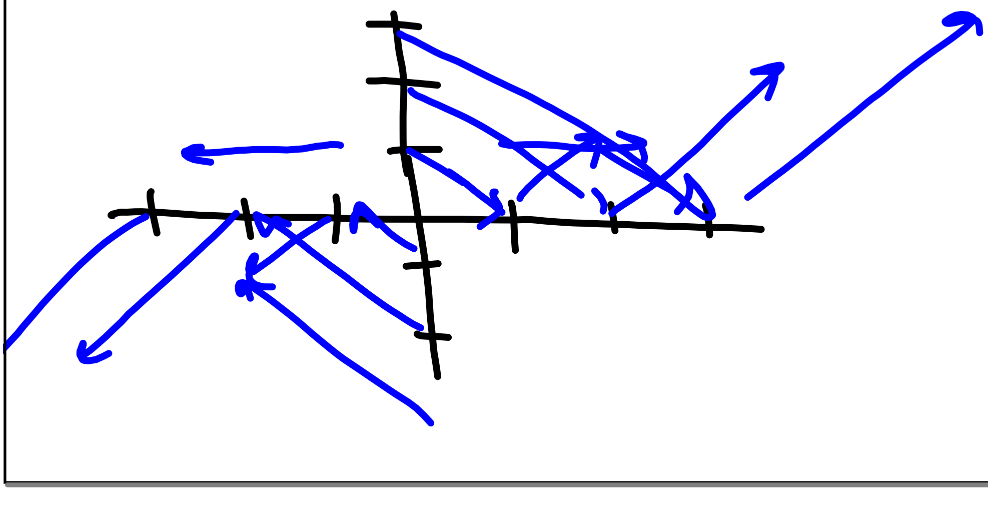
VECTOR FIELDS:

A VECTOR FIELD IN $\mathbb{R}^n$ ATTACHES A VECTOR $F(\underline{x})$ TO EACH POINT OF $\mathbb{R}^n$.

EXAMPLE: $F(x,y) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$



EXAMPLE:

$$f(x,y) = \frac{1}{2}x^2 + y$$

$$F(x,y) = \nabla f(x,y) = \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$g(x,y,z) = \frac{1}{4}(x^2+y^2+z^2)$$

$$G(x,y,z) = \nabla g = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \\ \frac{z}{2} \end{pmatrix}.$$

<u>THEOREM:</u> LET $f: \mathbb{R}^n \to \mathbb{R}$, DIFFERENTIABLE ON AN OPEN SET $D \subset \mathbb{R}^n$. AT EACH POINT $\underline{x}$ WHERE $\nabla f(\underline{x}) \neq 0$, ∇f POINTS IN THE DIRECTION OF MOST RAPID INCREASE.

<u>PROOF:</u> SEE LECTURE 12.

QUALITATIVELY, THE GRADIENT FIELD PICTURE DESCRIBES THE DIRECTION OF MOST RAPID INCREASE OF THE POTENTIAL FUNCTION.

CHAIN RULE:

EXAMPLE: GIVEN VECTOR FIELD

$f(t) = \begin{pmatrix} t \\ t^2 \\ t \end{pmatrix}$, AND FUNCTION

$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\cos(y+z)$

THE COMPOSITE FUNCTION

$g \circ f(t) = t\cos(t^2+t).$

$(g\circ f)'(t) = \cos(t^2+t)$
$\quad - t\sin(t^2+t)\cdot 2t.$

PROOF:

$$F'(t) = \lim_{h\to 0} \frac{F(t+h) - F(t)}{h}$$

$$= \lim_{h \to 0} \frac{g(f(t+h)) - g(f(t))}{h}.$$

BY THE MEAN VALUE THM FOR THE GRADIENT, FOR SOME x_0 ON THE SEGMENT BETWEEN x, y,

$$g(y) - g(x) = \nabla g(x_0) \cdot (y-x).$$

THUS, APPYING THIS WITH $y = f(t+h)$, $x = f(t)$

$$\frac{F(t+h) - F(t)}{h} = \nabla g(x_0) \frac{f(t+h) - f(t)}{h}.$$

LETTING $h \to 0$, $f(t+h) \to f(t)$

AND SO $x_0 \to f(t)$

$\implies \nabla g(x_0) \to \nabla g(f(t))$

AND $\dfrac{f(t+h) - f(t)}{h} \to f'(t).$

$\square$

A NORMAL VECTOR AT $\underline{x}_0 \in S_k$. TO THE LEVEL CURVE S_k IS A VECTOR $\underline{n}$ THAT IS PERPENDICULAR TO ANY SMOOTH CURVE IN S_k THRU $\underline{x}_0$..

<u>THEOREM:</u> LET $f: \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$ BE A POTENTIAL FUNCTION, AND S A LEVEL SET. THEN THE GRADIENT $\nabla f(\underline{x}_0)$ IS A NORMAL VECTOR TO THE SURFACE WHENEVER $\nabla f(x_0) \neq \underline{0}$.

EXAMPLE: $f(x,y) = x^3 + y^2$.

$S = \{ f(x,y) = 5 \}$.

$\underline{x}_0 = (1,2)$.

$\nabla f = \begin{pmatrix} 3x^2 \\ 2y \end{pmatrix}, \quad \nabla f(x_0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

THE TANGENT PLANE AT

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ IS GIVEN BY

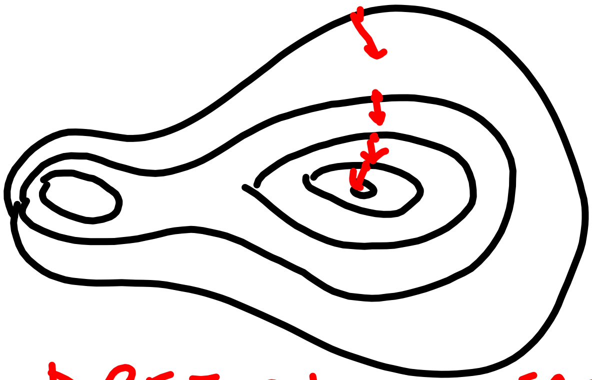
$$\nabla f(x_0) \cdot \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} = 0$$

OR $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} = 3x - 3 + 4y - 8$

$= 3x + 4y - 11$

$= 0$. $\square$

THEOREM: THE DIRECTION OF MAXIMUM INCREASE AT $\underline{x}_0$ IS PERPENDICULAR TO THE LEVEL SETS.



TOPOGRAPHICAL MAP. CURVES CONSTANT ELEVATION

→ DIRECTION OF GREATEST INCREASE

CHAIN RULE: RECALL, IF

$f: \mathbb{R}^n \to \mathbb{R}^m$ IS A DIFF. MAP,

$f'(x)$ IS A $m \times n$ MATRIX,

SUCH THAT

$$\lim_{\underline{y} \to \underline{0}} \frac{f(\underline{x}+\underline{y}) - f(\underline{x}) - f'(\underline{x})\underline{y}}{|\underline{y}|} = 0.$$

PROOF:

$$g'(f(x)) = \begin{pmatrix} \frac{\partial g_1}{\partial y_1}(f(x)) & \frac{\partial g_1}{\partial y_2}(f(x)) & \cdots & \frac{\partial g_1}{\partial y_m}(f(x)) \\ \vdots & & & \\ \frac{\partial g_p}{\partial y_1}(f(x)) & \frac{\partial g_p}{\partial y_2}(f(x)) & \cdots & \frac{\partial g_p}{\partial y_m}(f(x)) \end{pmatrix}$$

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}.$$

THE FIRST ROW OF THE PRODUCT IS GIVEN BY

$$\left(g'(f(x)) \cdot f'(x)\right)_1 = \left(\frac{\partial g_1}{\partial y_1}\frac{\partial f_1}{\partial x_1} + \frac{\partial g_1}{\partial y_2}\frac{\partial f_2}{\partial x_1} + \cdots + \frac{\partial g_1}{\partial y_m}\frac{\partial f_m}{\partial x_1}, \ \ldots \ \frac{\partial g_1}{\partial y_1}\frac{\partial f_1}{\partial x_n} + \cdots + \frac{\partial g_1}{\partial y_m}\frac{\partial f_m}{\partial x_n}\right).$$

THE iTH ENTRY OF THIS ROW IS GIVEN BY

$$\nabla g_1(f(x)) \cdot \frac{\partial f}{\partial x_i}.$$

THIS IS EXACTLY EQUAL TO

$\frac{\partial(g \circ f)}{\partial x_i}$. THIS PROVES THE CHAIN RULE,

FROM THE CHAIN RULE FOR $f(\gamma(t))$, γ A CURVE.

EXAMPLE:

$$w = g(u,v), \quad \begin{pmatrix} u \\ v \end{pmatrix} = f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

$$g \circ f(t) = g\left(f_1(t), f_2(t)\right).$$

$$\frac{d(g\circ f)}{dt} = \frac{dw}{dt} = \begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} f_1'(t) \\ f_2'(t) \end{pmatrix}$$

$$= \frac{\partial w}{\partial u} \cdot f_1'(t) + \frac{\partial w}{\partial v} f_2'(t).$$

EXAMPLE:

$$w = f(ax^2 + bxy + cy^2), \quad y = x^2+x+1.$$

CALCULATE: $\frac{dw}{dx}$.

$$\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \xrightarrow{f} \mathbb{R}$$

$$x \overset{g}{\longmapsto} \begin{pmatrix} x \\ x^2+x+1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{Q}{\longmapsto} ax^2+bxy+cy^2$$

$$\frac{d}{dx} f(ax^2+bxy+cy^2) = f'(ax^2+bxy+cy^2)\, Q'(x, x^2+x+1) \cdot g'(x).$$

$$Q' = [2ax + by,\ bx + 2cy].$$

$$g'(x) = \begin{pmatrix} 1 \\ 2x+1 \end{pmatrix}.$$

$$Q'(x, x^2+x+1) = [2ax + b(x^2+x+1),\ bx + 2c(x^2+x+1)]$$

FINAL PRODUCT:

$$f'(ax^2+bx(x^2+x+1)+c(x^2+x+1)^2) \cdot [2ax + b(x^2+x+1),\ bx + 2c(x^2+x+1)] \cdot \begin{bmatrix} 1 \\ 2x+1 \end{bmatrix}.$$