MAT 307 LECTURE 12

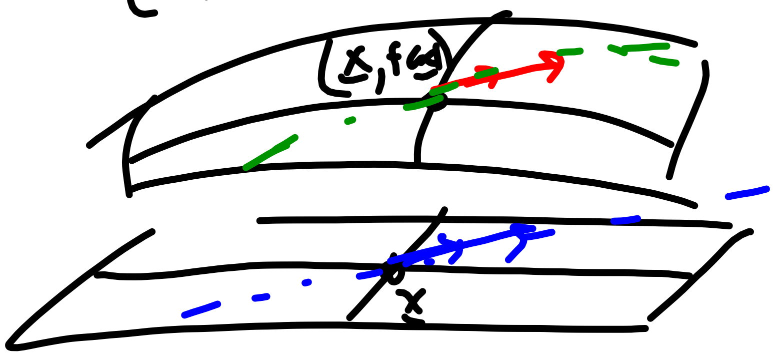
THE DIRECTIONAL DERIVATIVE

AND DERIVATIVE MATRIX

DEF'N: THE DERIVATIVE OF $f:\mathbb{R}^n \to \mathbb{R}$ IN DIRECTION $\underline{v}$ AT $\underline{x}$ IS

$$\frac{\partial f}{\partial \underline{v}} \text{ OR } D_{\underline{v}} f(\underline{x})$$

$$\frac{\partial f}{\partial \underline{v}} = \lim_{t \to 0} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x})}{t}$$



TREAT AS A SINGLE VARIABLE FN

ALONG THE LINE

$\ell(t) = \underline{x} + t\underline{v}$,

$f(\underline{x} + t\underline{v}) = g(t)$. CALCULATE $g'(0)$.

IN COORDINATES:

$$\frac{\partial f}{\partial \underline{v}}(\underline{x}) = v_1 \frac{\partial f}{\partial x_1}(\underline{x}) + v_2 \frac{\partial f}{\partial x_2}(\underline{x}) + \dots + v_n \frac{\partial f}{\partial x_n}(\underline{x}).$$

PROOF: ASSUME $\underline{v} \neq \underline{0}$, SINCE OTHERWISE, IF $\underline{v} = \underline{0}$,

$$\frac{\partial f}{\partial \underline{v}}(\underline{x}) = \lim_{t \to 0} \frac{f(\underline{x} + \underline{0}) - f(\underline{x})}{t} = 0.$$

IF THE SPECIAL CASE IS

$\underline{v} = \underline{e}_j$, j TH STANDARD BASIS VECTOR,

$$\lim_{t \to 0} \frac{f(\underline{x} + t\underline{e}_j) - f(\underline{x})}{t} = \frac{\partial f}{\partial x_j}(\underline{x})$$

$$= \frac{\partial f}{\partial \underline{e}_j}(\underline{x}).$$

IF $|\underline{v}| = 1$, $\frac{\partial f}{\partial \underline{v}}$ IS CALLED THE

<u>DIRECTIONAL DERIVATIVE</u>

IN DIRECTION $\underline{v}$.

PROOF:

$$\frac{\partial f}{\partial \underline{u}}(\underline{x}) = \nabla f(\underline{x}) \cdot \underline{u}.$$

THUS, BY CAUCHY-SCHWARZ,

$$\left| \frac{\partial f}{\partial \underline{u}}(\underline{x}) \right| \leq \| \nabla f(\underline{x}) \| \cdot \| \underline{u} \|.$$

IF $\|\underline{u}\| = 1$, THIS IS $\leq \|\nabla f(\underline{x})\|$.

THE EQUALITY CONDITION HOLDS IFF $\nabla f(\underline{x})$ AND $\underline{u}$ POINT IN THE SAME DIRECTION.

MAXIMUM IS $\underline{u} = \dfrac{\nabla f}{\|\nabla f\|}$

$$\Rightarrow D_{\underline{u}} f = \frac{\|\nabla f\|^2}{\|\nabla f\|} = \|\nabla f\|.$$

IF $\underline{u} = -\dfrac{\nabla f}{\|\nabla f\|}$ THEN

$$D_{\underline{u}} f = -\|\nabla f\|.$$

EXAMPLE: $f: \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = xyz.$

$$\nabla f = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}. \qquad \underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$D_{\underline{u}} f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \nabla f \cdot \underline{u} = u_1 yz + u_2 xz + u_3 xy.$$

RECALL THE MEAN VALUE THM FOR $f:[a,b] \to \mathbb{R}$.

THEOREM: (MEAN VALUE THEOREM)
SUPPOSE $f:[a,b] \to \mathbb{R}$ IS CTS ON $[a,b]$, DIFF. ON (a,b). THEN THERE EXISTS $c \in (a,b)$ SO THAT

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

PROOF: THE SEGMENT FROM $\underline{x}$ TO $\underline{y}$ IS TRAVERSED BY
$$m(t) = \underline{x} + t(\underline{y} - \underline{x}) \quad \text{FOR } t \in [0,1].$$

LET $g(t) = f(m(t))$, $t \in [0,1]$.

BY THE MEAN VALUE THEOREM, THERE IS $t_0 \in (0,1)$ SO THAT
$$f(\underline{y}) - f(\underline{x}) = g(1) - g(0)$$
$$= g'(t_0).$$

$$\frac{g(t+h) - g(t)}{h} = \frac{f(m(t) + h(\underline{y}-\underline{x})) - f(m(t))}{h}$$

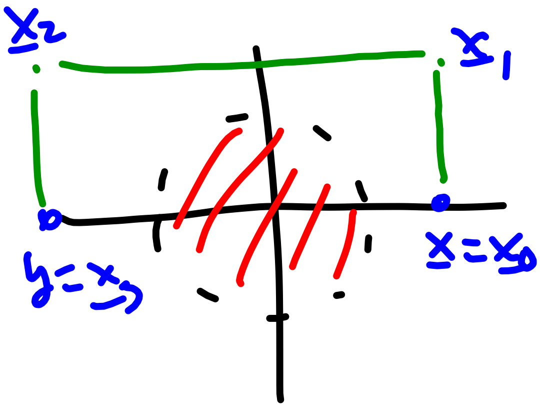
So
$$g'(t) = D_{\underline{y}-\underline{x}} f(m(t))$$
$$= \nabla f(m(t)) \cdot (\underline{y} - \underline{x}).$$

THUS $f(\underline{y}) - f(\underline{x}) = \nabla f(m(t_0)) \cdot (\underline{y} - \underline{x})$.

LET $\underline{x}_0 = m(t_0) = \underline{x} + t_0(\underline{y} - \underline{x})$.

$\square$

EXAMPLE: $\mathbb{R}^2 \setminus \{(x,y) : x^2+y^2<1\}$.



PROOF: FIX A POINT $\underline{x}_0 \in U$. LET $c = f(\underline{x}_0)$.

GIVEN ANY OTHER POINT $\underline{y} \in U$, CHOOSE A POLYGONAL PATH $\underline{x}_0, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n = \underline{y}$ IN U. THEN

$$f(\underline{x}_i) - f(\underline{x}_{i-1}) = \nabla f(\underline{x}) \cdot (\underline{x}_i - \underline{x}_{i-1})$$

FOR SOME $\underline{x}$ ON THE SEGMENT FROM x_i TO x_{i-1}

$$= 0.$$

$$\Rightarrow f(\underline{x}_i) = f(\underline{x}_{i-1}) = c.$$

SO $f(\underline{y}) = c$, ALL $\underline{y}$. $\square$

IF ALL PARTIAL DERIVATIVES OF COORDINATE FUNCTIONS ARE CONTINUOUS, WE SAY f IS <u>CONTINUOUSLY DIFFERENTIABLE</u>.

EXAMPLE: $f: \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x,y) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

$$\nabla f_1(x,y) = (2x, -2y)$$

$$\nabla f_2(x,y) = (2y, 2x)$$

EXAMPLE: $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x,y,z) = \begin{pmatrix} x^2 + e^y \\ x + y\sin z \\ x + y \end{pmatrix}$$

$$f'(x,y,z) = \begin{pmatrix} 2x & e^y & 0 \\ 1 & \sin z & y\cos z \\ 1 & 1 & 0 \end{pmatrix}$$

EXAMPLE: $f: \mathbb{R} \to \mathbb{R}^2$

$$f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$f'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

EXAMPLE: $f: \mathbb{R}^2 \to \mathbb{R}^3$,

$$f(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ u \end{pmatrix}$$

$$f'(u,v) = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 1 & 0 \end{pmatrix}.$$

AT $(1,0)$, $f(1,0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$f'(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T(x,y) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y \end{pmatrix}.$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x-1 \\ y \\ x-1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ x \end{pmatrix}.$$

THEOREM: IF $f: \mathbb{R}^n \to \mathbb{R}^m$ IS DIFFERENTIABLE AT $\underline{x}_0$ THEN

$$\lim_{\underline{x} \to \underline{x}_0} \frac{f(\underline{x}) - f(\underline{x}_0) - f'(\underline{x}_0)(\underline{x} - \underline{x}_0)}{\| \underline{x} - \underline{x}_0 \|} = 0$$

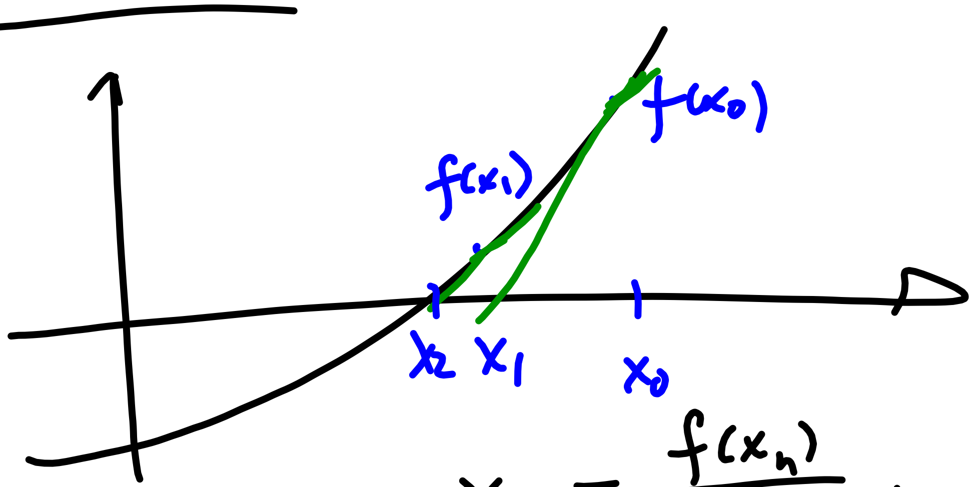
AND $f'(\underline{x}_0)$ IS THE UNIQUE MATRIX WHICH SATISFIES THIS.

COROLLARY: IF $f: \mathbb{R}^n \to \mathbb{R}^m$
$f(\underline{x}) = A \cdot \underline{x}$, THEN
$f'(\underline{x}) = A$.

PROOF: f' IS UNIQUE,

BUT $$\lim_{\underline{x} \to \underline{x}_0} \frac{A(\underline{x}) - A(\underline{x}_0) - A(\underline{x} - \underline{x}_0)}{\|\underline{x} - \underline{x}_0\|} = 0. \quad \square$$

RECALL: NEWTON'S METHOD FOR $f: \mathbb{R} \to \mathbb{R}$ TO LOCATE A ZERO:



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

USED TO CALCULATE:

1. EIGENVALUES $M\underline{v} = \lambda \underline{v}$
   $\iff (M - \lambda)\underline{v} = \underline{0}$.

2. FIXED POINTS $f(\underline{x}) = \underline{x}$
   $\iff G(\underline{x}) = 0$

   $G(\underline{x}) = f(\underline{x}) - \underline{x}$.

ETC.