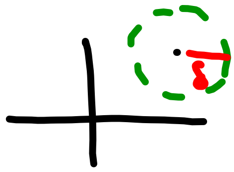
RECALL FROM LAST LECTURE:

THE δ- NEIGHBORHOOD OF A POINT $\underline{x}_0$



$$B_\delta(\underline{x}_0) = \{\underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{x}_0\| < \delta\}$$

DEFINITION: WE SAY A FUNCTION $f: \mathbb{R}^n \to \mathbb{R}^m$ HAS LIMIT $\underline{y}_0 \in \mathbb{R}^m$ AT $\underline{x}_0 \in \mathbb{R}^n$ IF, FOR ANY $\varepsilon > 0$ THERE EXISTS $\delta > 0$ SO THAT IF

$$0 < \|\underline{x} - \underline{x}_0\| < \delta$$

THEN $\|f(\underline{x}) - \underline{y}_0\| < \varepsilon$.

REMARK: THIS IS THE SAME DEFINITION AS FOR FNS ON $\mathbb{R}$, WITH $\|\cdot\|$ PLAYING ROLE OF ABS. VALUE.

<u>THEOREM</u>: GIVEN $f: \mathbb{R}^n \to \mathbb{R}^m$, WITH COORDINATE FUNCTIONS $f_1(\underline{x}), \ldots, f_m(\underline{x})$, WE HAVE

$$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = \underline{y}$$

IF AND ONLY IF, FOR EACH $i = 1, 2, \ldots, m$, $\lim_{\underline{x} \to \underline{x}_0} f_i(\underline{x}) = y_i$.

GIVEN $\varepsilon > 0$, CHOOSE $\delta_i > 0$ SO THAT IF

$0 < \|\underline{x} - \underline{x}_0\| < \delta_i$ THEN

$\| f_i(\underline{x}) - y_i \| < \frac{\varepsilon}{\sqrt{m}}$.

LET $\delta = \min(\delta_1, \dots, \delta_m)$.

THEN IF $0 < \|\underline{x} - \underline{x}_0\| < \delta$, FOR EACH $i = 1, 2, \dots, m$,

$|f_i(\underline{x}) - y_i| < \frac{\varepsilon}{\sqrt{m}}$.

HENCE $\sqrt{(f_1(x) - y_1)^2 + \dots + (f_m(x) - y_m)^2}$

$< \sqrt{m\left(\frac{\varepsilon^2}{m}\right)} = \varepsilon$.

THUS $\| f(\underline{x}) - \underline{y} \| < \varepsilon$. $\square$

A POINT $x_0 \in D$ WHICH IS NOT A LIMIT POINT IS CALLED AN ISOLATED POINT.

BY CONVENTION WE SAY A FUNCTION IS CONTINUOUS AT ANY ISOLATED POINT.

Ex: $\mathbb{Z} \subset \mathbb{R}$ ALL ARE ISOLATED POINTS.

DEFINE THE kTH COORDINATE PROJECTION

$$p_k : \mathbb{R}^n \to \mathbb{R}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad p_k(\underline{x}) = x_k.$$

<u>THEOREM</u>: p_k IS CONTINUOUS.

<u>PF</u>: GIVEN $\varepsilon > 0$ CHOOSE $\delta = \varepsilon$.

IF $\|\underline{x} - \underline{x}_0\| < \delta$ THEN

$|x_1 - x_{0,1}|, \ldots, |x_n - x_{0,n}| < \delta$

$\Rightarrow |p_k(\underline{x}) - p_k(\underline{x}_0)| < \varepsilon = \delta.$ $\square$

$M(x_1, y_1) = x_1 y_1$

$M(x_2, y_2) = x_2 y_2$.

$$x_1 y_1 - x_2 y_2 = x_1 y_1 - x_1 y_2 + x_1 y_2 - x_2 y_2$$
$$= x_1 (y_1 - y_2) + y_2 (x_1 - x_2).$$

GIVEN THE POINT (x_1, y_1) AND $\varepsilon > 0$, IF WE ASSUME $\|(x_2, y_2) - (x_1, y_1)\| < \delta$ FOR SOME $\delta < 1$

THEN

$$|M(x_1, y_1) - M(x_2, y_2)| < |x_1| \cdot \underbrace{|y_1 - y_2|}_{<\delta} + (|y_2|) |x_1 - x_2|$$

WE HAVE $|y_2| \leq |y_1| + \delta$.

$$\leq |x_1| \cdot \delta + (|y_1| + \delta) \cdot \delta$$
$$\leq |x_1| \cdot \delta + (|y_1| + 1) \cdot \delta.$$

IF WE CHOOSE

$$\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{|x_1| + |y_1| + 1}\right)$$

THEN IF $\|(x_1, y_1) - (x_2, y_2)\| < \delta$

WE HAVE $|M(x_1, y_1) - M(x_2, y_2)| < \varepsilon$. $\square$

THEOREM: IF $f: \mathbb{R}^m \to \mathbb{R}^n$ IS LINEAR, f IS CONTINUOUS.

PROOF: BY OUR DISCUSSION FROM LINEAR ALGEBRA, THERE IS AN $n \times m$ MATRIX A,
$$\begin{pmatrix} - & \underline{\alpha}_1 & - \\ - & \underline{\alpha}_2 & - \\ & \vdots & \\ - & \underline{\alpha}_n & - \end{pmatrix}$$
SO THAT
$$f(\underline{x}) = A \cdot \underline{x} = \begin{pmatrix} \underline{\alpha}_1 \cdot \underline{x} \\ \underline{\alpha}_2 \cdot \underline{x} \\ \vdots \\ \underline{\alpha}_n \cdot \underline{x} \end{pmatrix}$$

$$f(\underline{x}) - f(\underline{y}) = f(\underline{x} - \underline{y}).$$

$$\|f(\underline{v})\| = \sqrt{(\underline{\alpha}_1 \cdot \underline{v})^2 + \dots + (\underline{\alpha}_n \cdot \underline{v})^2}$$

BY CAUCHY-SCHWARZ, $|\alpha_i \cdot v|^2 \le \|\underline{\alpha}_i\|^2 \cdot \|\underline{v}\|^2$.

$$\|f(\underline{v})\| \le \sqrt{(\|\underline{\alpha}_1\|^2 + \dots + \|\underline{\alpha}_n\|^2)\|\underline{v}\|^2}$$
$$= \|\underline{v}\| \cdot \sqrt{\|\underline{\alpha}_1\|^2 + \dots + \|\underline{\alpha}_n\|^2}.$$

THUS, IF $\|\underline{x} - \underline{y}\| < \dfrac{\varepsilon}{\sqrt{\|\underline{\alpha}_1\|^2 + \dots + \|\underline{\alpha}_n\|^2} + 1}$

THEN $\|f(\underline{x}) - f(\underline{y})\| < \varepsilon$.

SO f IS CTS. $\square$

DEFINITION: A FUNCTION

$f : \mathbb{R}^n \to \mathbb{R}$

IS DIFFERENTIABLE AT $\underline{x}_0 \in \mathrm{Dom}(f)$

IF

(i) $\underline{x}_0$ IS AN INTERIOR PT OF f.

(ii) THERE IS A VECTOR $\underline{a}$,

SO THAT

$$\lim_{\underline{x} \to \underline{x}_0} \frac{f(\underline{x}) - f(\underline{x}_0) - \underline{a} \cdot (\underline{x} - \underline{x}_0)}{\|\underline{x} - \underline{x}_0\|} = \underline{0}.$$

RECALL f DIFF AT $\underline{x}_0$

$f : \mathbb{R}^n \to \mathbb{R}$ $\iff$ THERE IS A VECTOR $\underline{a} \in \mathbb{R}^n$,

$$\lim_{\underline{x} \to \underline{x}_0} \frac{f(\underline{x}) - f(\underline{x}_0) - \underline{a} \cdot (\underline{x} - \underline{x}_0)}{|\underline{x} - \underline{x}_0|} = 0$$

THE VECTOR $\underline{a}$ IS CALLED THE GRADIENT, WRITTEN

$\underline{a} = \nabla f(\underline{x}_0)$. ∇ = "NABLA"

PROOF:

RECALL $\lim_{\underline{x} \to \underline{x}_0} \dfrac{f(\underline{x}) - f(\underline{x}_0) - \nabla f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0)}{|\underline{x} - \underline{x}_0|} = 0.$

CHOOSE $\underline{x} = \underline{x}_0 + t\,\underline{e}_j$,

$\underline{e}_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← jTH SLOT.

LET $t \to 0$.

THEN $|\underline{x} - \underline{x}_0| = |t|$

$\nabla f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0) = t \cdot \nabla f(\underline{x}_0)_j$.

$$\lim_{t \to 0} \frac{f(\underline{x}_0 + t\,\underline{e}_j) - f(\underline{x}_0) - t\,\nabla f(\underline{x}_0)_j}{|t|} = 0.$$

THUS $\frac{\partial}{\partial x_j} f(\underline{x}_0) = \nabla f(\underline{x}_0)_j$. $\square$

THM: IF $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ IS DIFF. AT $\underline{x}_0$ THEN f IS CTS AT $\underline{x}_0$.

PROOF: WE HAVE

$$\lim_{\underline{x} \to \underline{x}_0} \frac{f(\underline{x}) - f(\underline{x}_0) - \nabla f(\underline{x}_0)(\underline{x} - \underline{x}_0)}{|\underline{x} - \underline{x}_0|} = 0.$$

PROD OF TWO QUANTITIES $\to 0$.

$$f(\underline{x}) - f(\underline{x}_0) = |\underline{x} - \underline{x}_0| \cdot \left(\frac{f(\underline{x}) - f(\underline{x}_0) - \nabla f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0)}{|\underline{x} - \underline{x}_0|} \right) + \nabla f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0).$$

AS $\underline{x} \to \underline{x}_0$, $\nabla f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0)$ IS LINEAR SO TENDS TO 0.

THUS $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) - f(\underline{x}_0) = \underline{0}$.

IN HIGHER DIMENSIONS,
IF $f: \mathbb{R}^n \to \mathbb{R}$, THE TANGENT PLANE TO THE GRAPH $\begin{pmatrix} \underline{x} \\ f(\underline{x}) \end{pmatrix}$ AT $\begin{pmatrix} \underline{x}_0 \\ f(\underline{x}_0) \end{pmatrix}$

IS GIVEN BY THE BEST LINEAR APPROX

$$\begin{pmatrix} \underline{x} \\ f(\underline{x}_0) + \nabla f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0) \end{pmatrix}$$

EXAMPLE: $f(x,y,z) = xy^2z^3$

$$\nabla f = \begin{pmatrix} y^2z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{pmatrix}.$$

$$\nabla f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

THE BEST LINEAR APPROX TO f AT $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ IS $f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix}$

$$= \underline{1} + (x-1) + 2(y-1) + 3(z-1)$$

$$= x + 2y + 3z - 5.$$