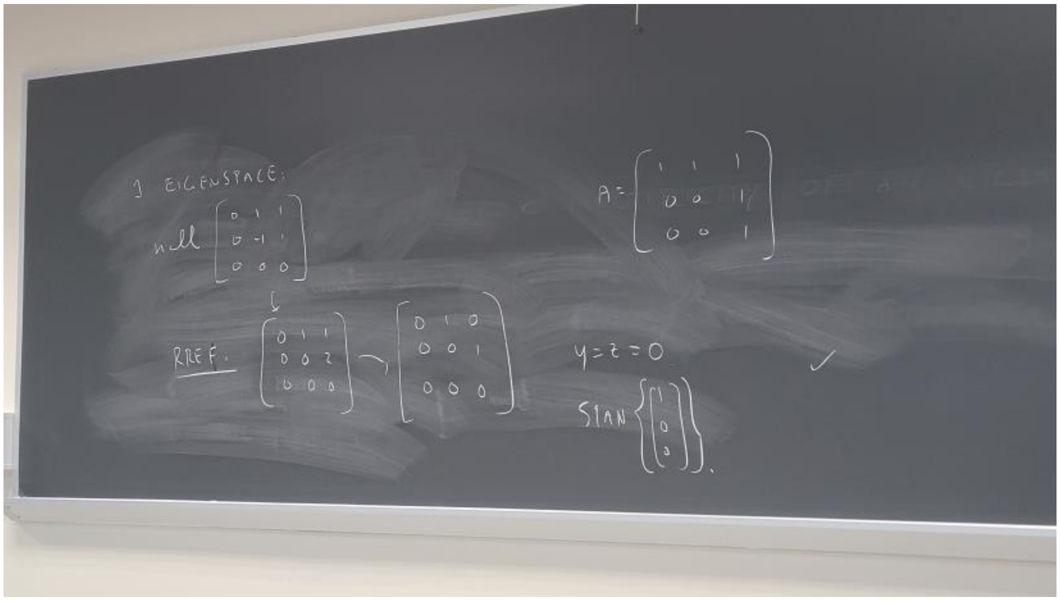
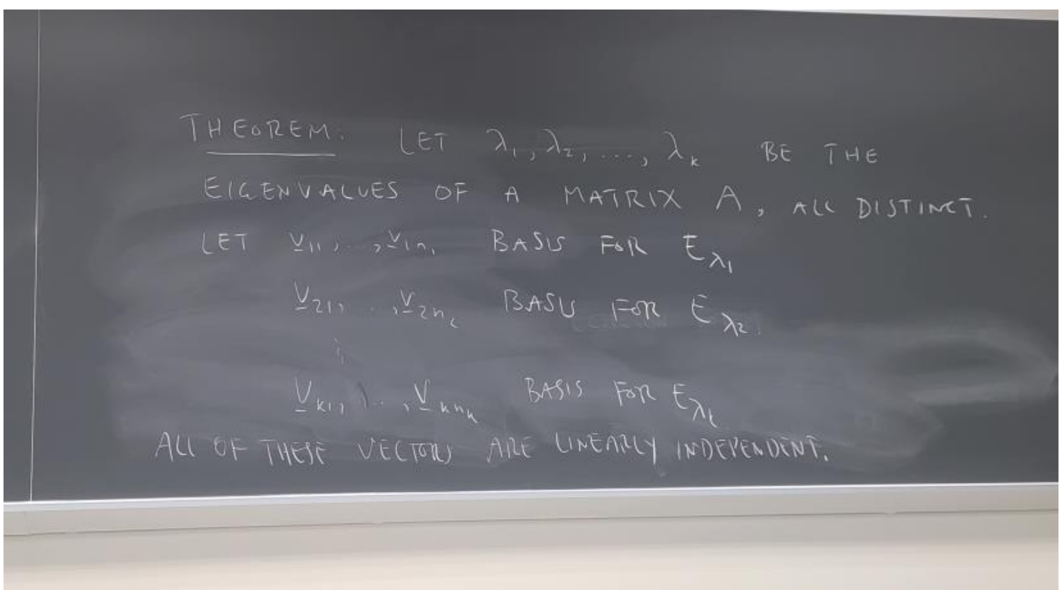
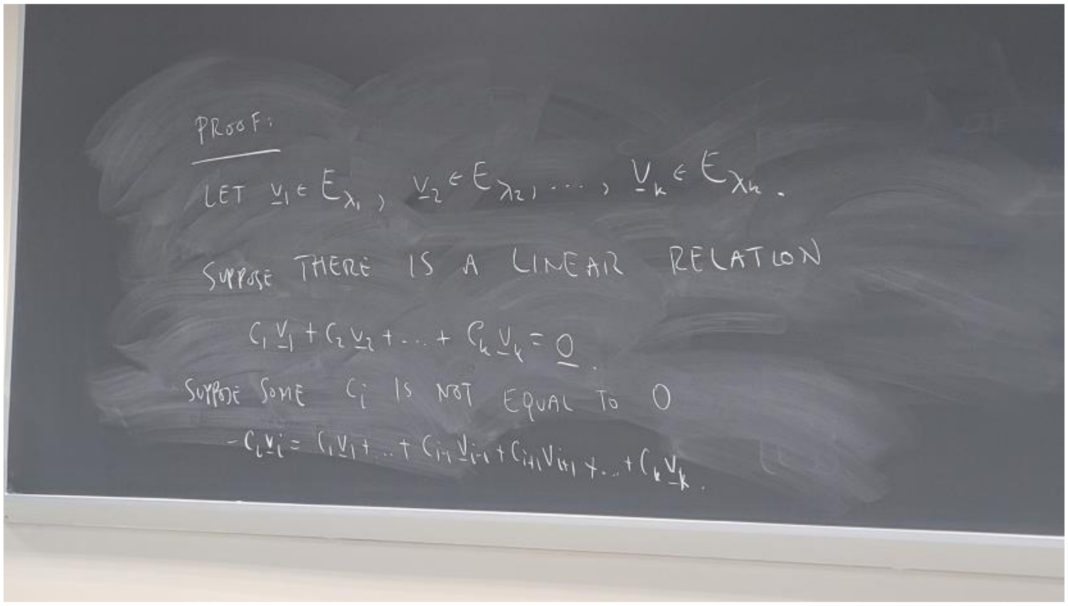
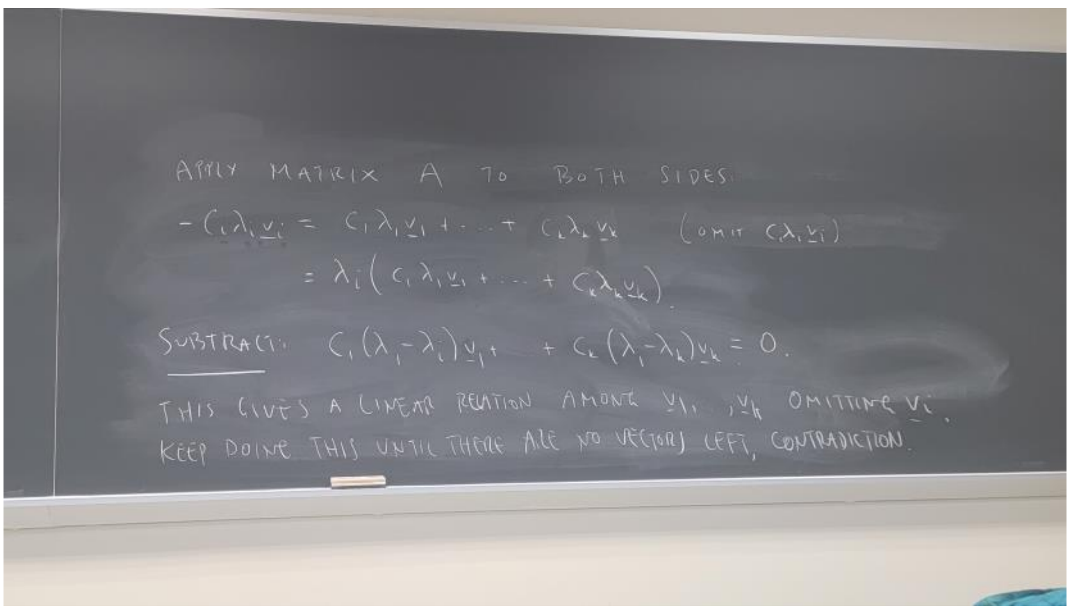
4/6/23

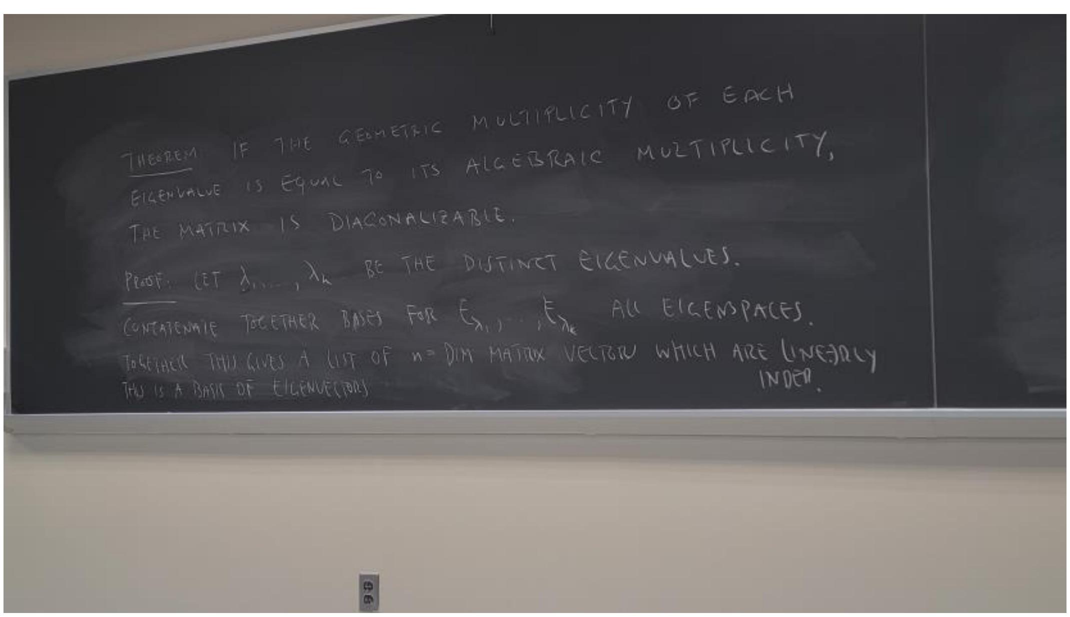
Saturday, April 8, 2023 5:38 PM

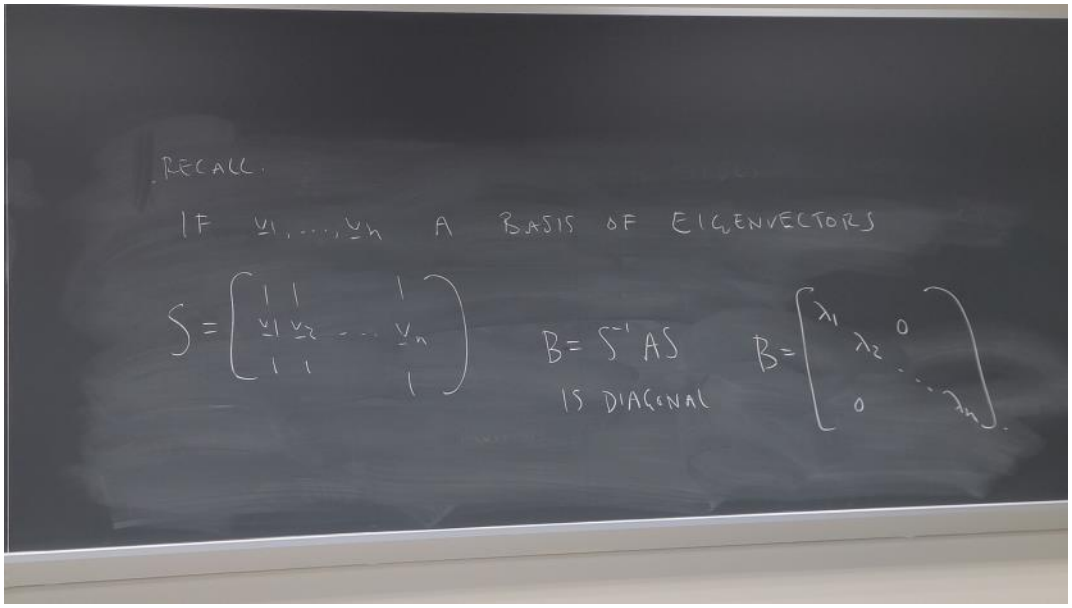


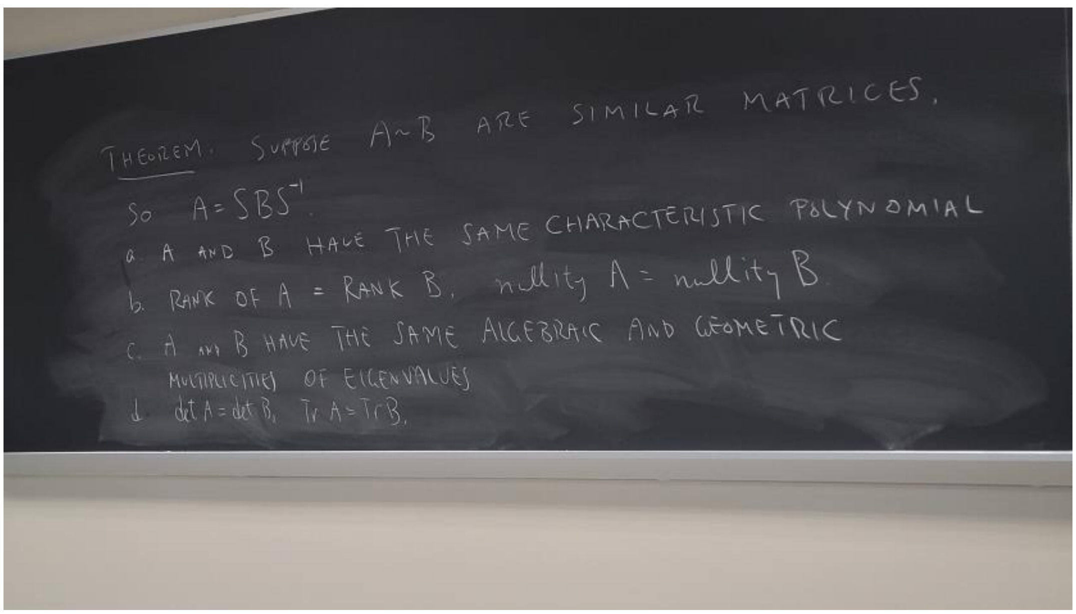


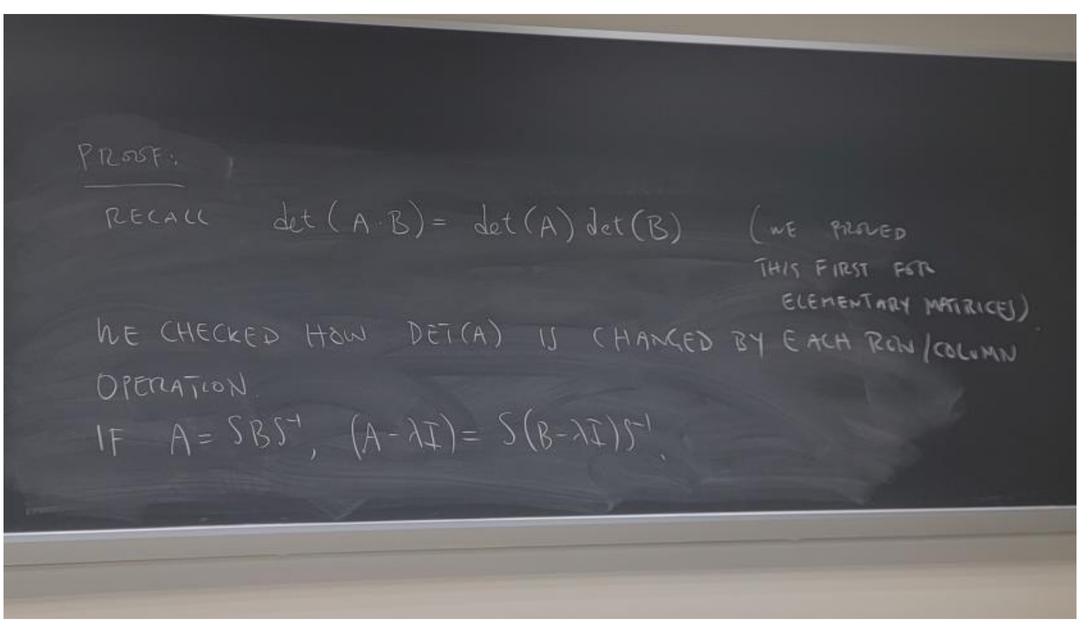

PROOF:
LET $\underline{v}_1 \in E_{\lambda_1}$, $\underline{v}_2 \in E_{\lambda_2}, \dots, \underline{v}_k \in E_{\lambda_k}$.
SUPPOSE THERE IS A LINEAR RELATION
$c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k = \underline{0}$.
SUPPOSE SOME c_i IS NOT EQUAL TO 0
$-c_i\underline{v}_i = c_1\underline{v}_1 + \dots + c_{i-1}\underline{v}_{i-1} + c_{i+1}v_{i+1} + \dots + c_k\underline{v}_k$.


APPLY MATRIX A TO BOTH SIDES.
$-c_i\lambda_i\underline{v}_i = c_1\lambda_1\underline{v}_1 + \dots + c_k\lambda_k\underline{v}_k$ (OMIT $c_i\lambda_i\underline{v}_i$)
$= \lambda_i(c_1\lambda_1\underline{v}_1 + \dots + c_k\lambda_k\underline{v}_k)$.
SUBTRACT: $c_1(\lambda_1-\lambda_i)\underline{v}_1 + \dots + c_k(\lambda_1-\lambda_k)\underline{v}_k = 0$.
THIS GIVES A LINEAR RELATION AMONG $\underline{v}_1, \dots, \underline{v}_k$ OMITTING $\underline{v}_i$.
KEEP DOING THIS UNTIL THERE ARE NO VECTORS LEFT, CONTRADICTION.


THEOREM: IF THE GEOMETRIC MULTIPLICITY OF EACH EIGENVALUE IS EQUAL TO ITS ALGEBRAIC MULTIPLICITY, THE MATRIX IS DIAGONALIZABLE.
PROOF: LET $\lambda_1, \dots, \lambda_k$ BE THE DISTINCT EIGENVALUES.
CONCATENATE TOGETHER BASES FOR $E_{\lambda_1}, \dots, E_{\lambda_k}$ ALL EIGENSPACES.
TOGETHER THIS GIVES A LIST OF $n = $ DIM MATRIX VECTORS WHICH ARE LINEARLY INDEP.
THIS IS A BASIS OF EIGENVECTORS.


RECALL.
IF $\underline{v}_1, \dots, \underline{v}_n$ A BASIS OF EIGENVECTORS
$S = \begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{pmatrix}$
$B = S^{-1}AS$
IS DIAGONAL
$B = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$.


THEOREM. SUPPOSE $A \sim B$ ARE SIMILAR MATRICES,
SO $A = SBS^{-1}$.
a. A AND B HAVE THE SAME CHARACTERISTIC POLYNOMIAL
b. RANK OF A = RANK B, nullity A = nullity B.
c. A AND B HAVE THE SAME ALGEBRAIC AND GEOMETRIC
MULTIPLICITIES OF EIGENVALUES
d. $\det A = \det B$, $\operatorname{Tr} A = \operatorname{Tr} B$.


PROOF:
RECALL $\det(A \cdot B) = \det(A)\det(B)$
(WE PROVED
THIS FIRST FOR
ELEMENTARY MATRICES)
WE CHECKED HOW DET(A) IS CHANGED BY EACH ROW/COLUMN
OPERATION.
IF $A = SBS^{-1}$, $(A - \lambda I) = S(B - \lambda I)S^{-1}$.

$$f_A(\lambda) = \det(A - \lambda I)$$
$$= \det\left(S(B - \lambda I)S^{-1}\right)$$
$$= \cancel{\det S}\ \det(B - \lambda I)\ \cancel{\det(S^{-1})}$$
$$= \det(B - \lambda I) = f_B(\lambda).$$

b. IF $B\underline{x} = \underline{0}$, $y = Sx$. $\quad Ay = SBS^{-1}y = SBS^{-1}Sx = SBx = S \cdot 0 = 0.$

IF $Ay = 0$, $x = S^{-1}y$ $\quad B\underline{x} = S^{-1}AS\underline{x} = S^{-1}ASS^{-1}y = S^{-1}Ay = S^{-1}\underline{0} = 0.$

IT FOLLOWS THAT IF

$\underline{x}_1, \dots, \underline{x}_k$ ARE A BASIS FOR NULL (B).

$S\underline{x}_1, \dots, S\underline{x}_k$ ARE LINEARLY INDEPENDENT IN null(A).

IF $y_1, \dots, y_\ell$ ARE A BASIS FOR NULL(A)

$S^{-1}y_1, \dots, S^{-1}y_\ell$ ARE L.I. IN null(B).

$\Rightarrow$ nullity(A) = nullity(B). SINCE RANK + NULLITY = n, RANK A = RANK B.

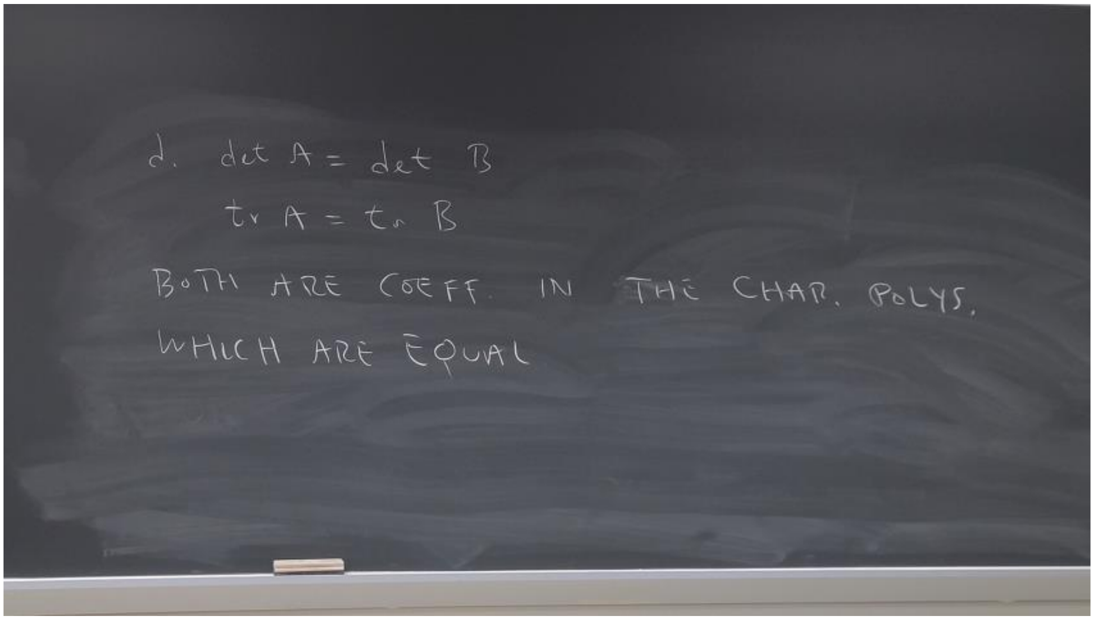
c. THE ALGEBRAIC MULTIPLICITIES ARE EQUAL BECAUSE THE MATRICES HAVE THE SAME CHAR POLY.

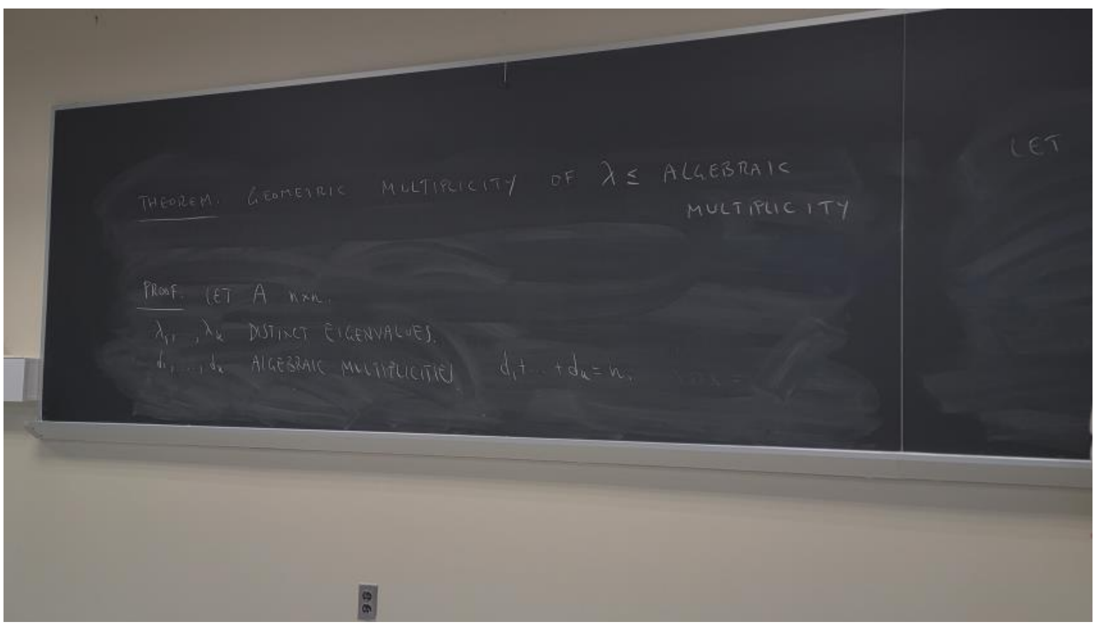
IF $A = SBS^{-1}$

$(A - \lambda I) = S(B - \lambda I)S^{-1}$.

GEOM MULT OF λ = nullity$(A - \lambda I)$

= nullity$(B - \lambda I)$.


d. $\det A = \det B$
$\operatorname{tr} A = \operatorname{tr} B$
BOTH ARE COEFF. IN THE CHAR. POLYS.
WHICH ARE EQUAL


THEOREM. GEOMETRIC MULTIPLICITY OF $\lambda \leq$ ALGEBRAIC MULTIPLICITY
PROOF. LET A $n \times n$.
$\lambda_1, \ldots, \lambda_k$ DISTINCT EIGENVALUES.
$d_1, \ldots, d_k$ ALGEBRAIC MULTIPLICITIES $d_1 + \ldots + d_k = n$.
LET

LET $e_1, \dots, e_k$ GEOMETRIC MULTIPLICITIES.

$\underline{v}_{11}, \dots, \underline{v}_{1e_1}$ BASIS FOR $null(A - \lambda_1 I)$

$\underline{v}_{21}, \dots, \underline{v}_{2e_2}$ BASIS FOR $null(A - \lambda_2 I)$

$\vdots$

$\underline{v}_{k1}, \dots, \underline{v}_{ke_k}$ BASIS FOR $null(A - \lambda_k I)$.

$\underline{v}_{11}, \dots, \underline{v}_{1e_1}, \underline{v}_{21}, \dots, \underline{v}_{2e_2}, \dots, \underline{v}_{k1}, \dots, \underline{v}_{ke_k}$ LINEARLY INDEPENDENT.

ADD VECTORS $\underline{w}_1, \dots, \underline{w}_l$ TO REACH A BASIS.

WITH RESPECT TO THIS BASIS THE MATRIX BECOMES

$$Q = \left[\begin{array}{ccc|c} \underbrace{\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}}_{e_1} & 0 & & \\ & \underbrace{\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}}_{e_2} & 0 & C \\ & & \underbrace{\begin{matrix}\lambda_k & 0 \\ 0 & \lambda_k\end{matrix}}_{e_k} & \\ \hline & 0 & & B \end{array}\right]$$

ONLY ONE NON-ZERO ENTRY IN COLUMNS $1 \to e_1 + \dots + e_k$. MUST SELECT WHEN CALCULATING det.

$$\det(Q - \lambda I) = (\lambda_1 - \lambda)^{e_1} \cdots (\lambda_k - \lambda)^{e_k} \det(B - \lambda I).$$

$\det(A - \lambda I)$ "="

THIS PROVES $e_1 \leq d_1, e_2 \leq d_2, \dots, e_k \leq d_k$ $\square$

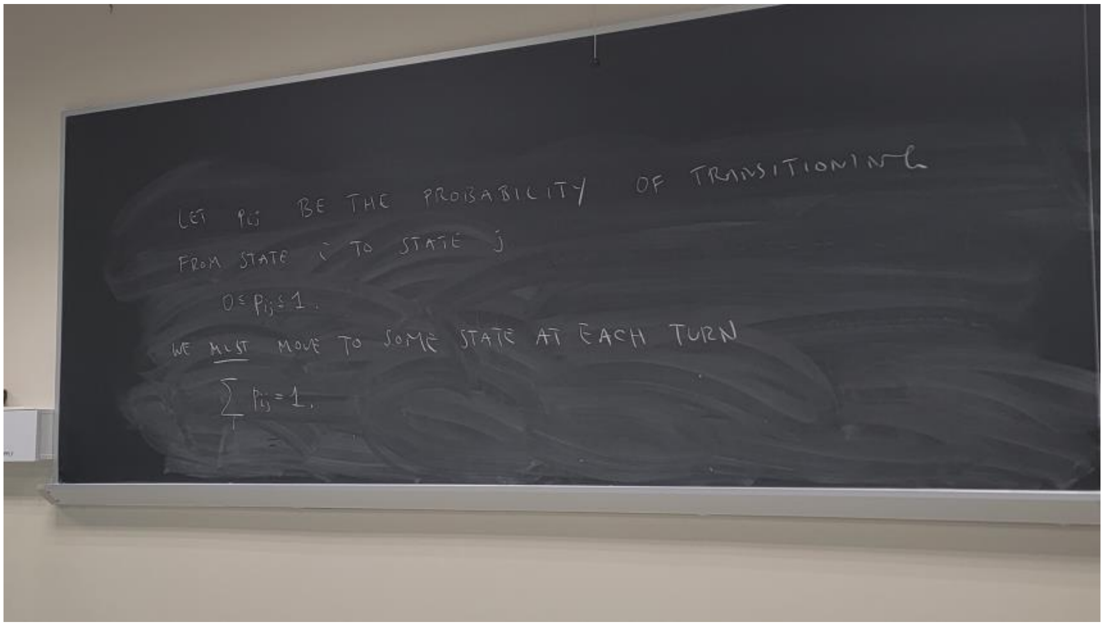
APPLICATION: MARKOV CHAIN

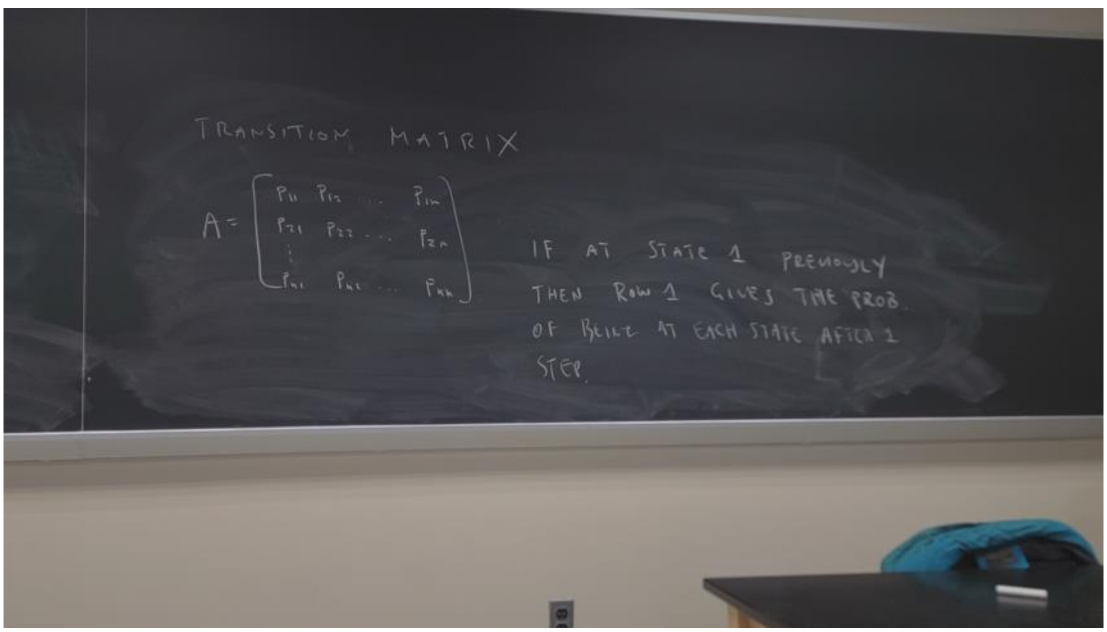
FINITE STATE MARKOV CHAINS.

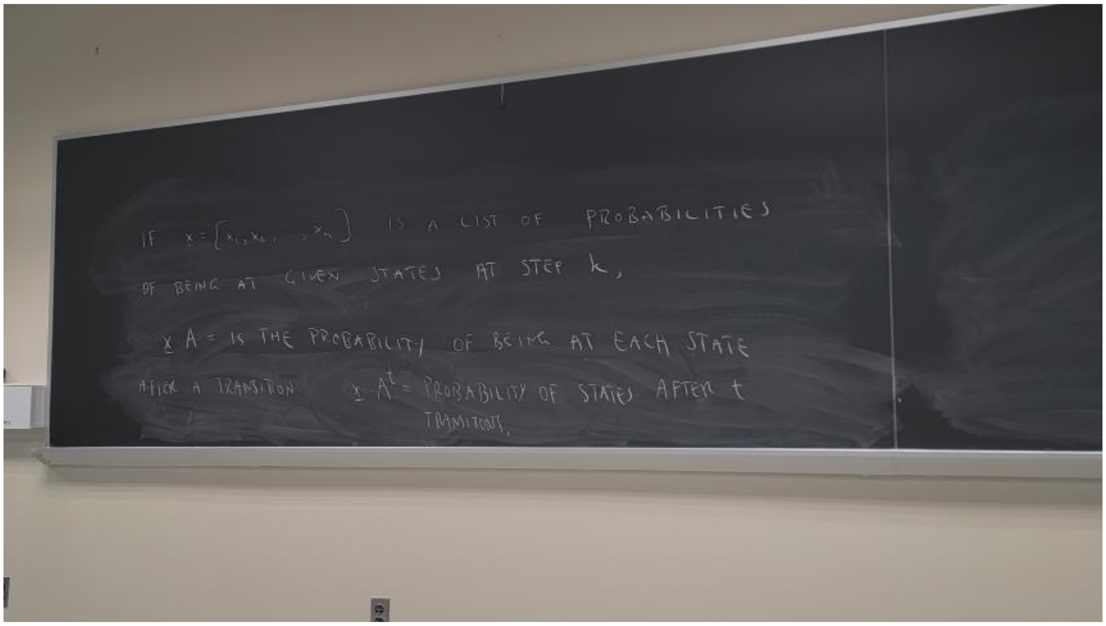
GIVEN A SYSTEM WHICH CAN BE IN ANY OF THE STATES

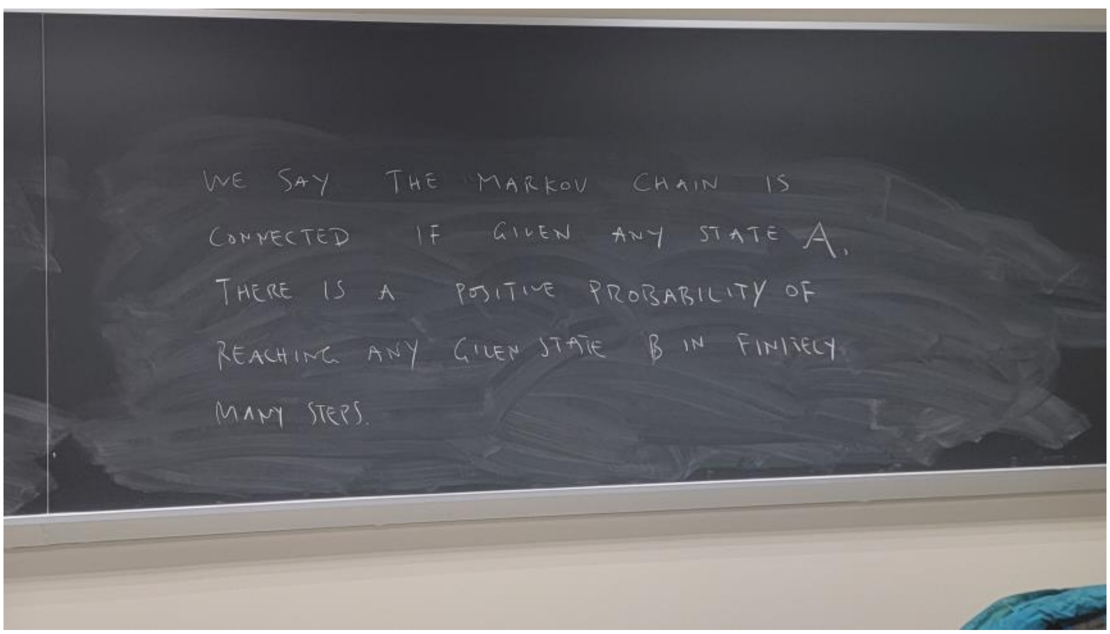
$\underline{A}_1, \underline{A}_2, \underline{A}_3, \dots, \underline{A}_n$

AT A GIVEN STEP, THE SYSTEM MOVES FROM ONE STATE TO ANOTHER AT RANDOM.


LET p_{ij} BE THE PROBABILITY OF TRANSITIONING
FROM STATE i TO STATE j
$0 \le p_{ij} \le 1$.
WE MUST MOVE TO SOME STATE AT EACH TURN
$\sum_j p_{ij} = 1$.


TRANSITION MATRIX
$A = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}$
IF AT STATE 1 PREVIOUSLY
THEN ROW 1 GIVES THE PROB.
OF BEING AT EACH STATE AFTER 1
STEP.


IF x = [x₁, x₂, …, xₙ] IS A LIST OF PROBABILITIES
OF BEING AT GIVEN STATES AT STEP k,
x·A = IS THE PROBABILITY OF BEING AT EACH STATE
AFTER A TRANSITION.
x·A^t = PROBABILITY OF STATES AFTER t
TRANSITIONS.


WE SAY THE MARKOV CHAIN IS
CONNECTED IF GIVEN ANY STATE A,
THERE IS A POSITIVE PROBABILITY OF
REACHING ANY GIVEN STATE B IN FINITELY
MANY STEPS.

<u>THEOREM:</u> LET A BE THE TRANSITION MATRIX OF A CONNECTED MARKOV CHAIN. THE LARGEST EIGENVALUE OF A IS 1. THERE IS A (LEFT) EIGENVECTOR WITH POSITIVE ENTRIES, EIGENVALUE 1, CALLED THE <u>STATIONARY DISTRIBUTION</u>.

ANY OTHER LEFT EIGENVECTOR HAS EIGENVALUE < 1.

NOTICE $\det(A - \lambda I)$

$$= \det(A^t - \lambda I)$$

ALG. MULT. OF EIGENVALUES OF A, A^t ARE THE SAME.

PROOF: LET $\underline{x}$ BE A LEFT EIGENVECTOR,

$$\underline{x} = [x_1\ x_2 \ldots x_n]$$

$$\underline{x} \cdot A = \lambda \underline{x}.$$

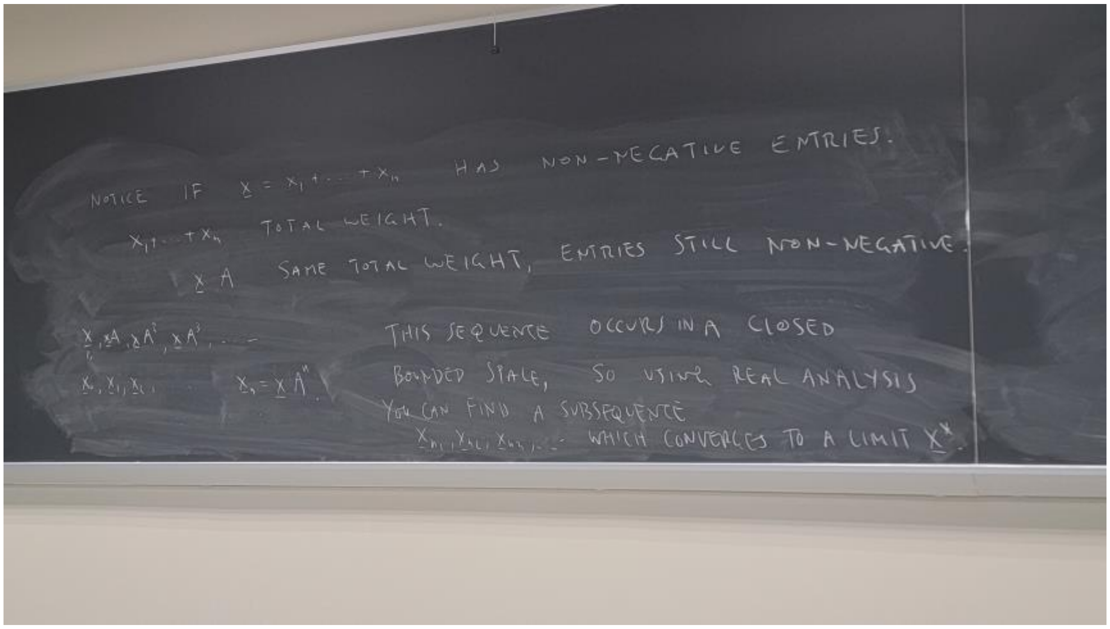
$$\|\underline{x}\|_1 = |x_1| + |x_2| + \ldots + |x_n|.$$

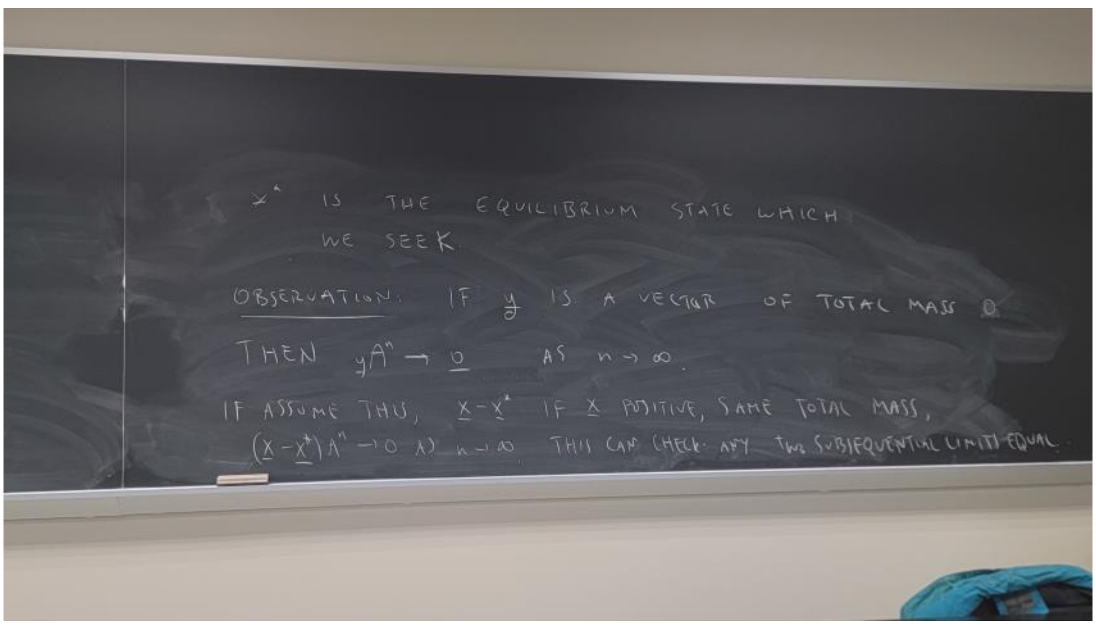
$$\|\underline{x}A\|_1 = |\lambda x_1| + \cdots + |\lambda x_n| = |\lambda| \left(|x_1| + \ldots + |x_n| \right).$$

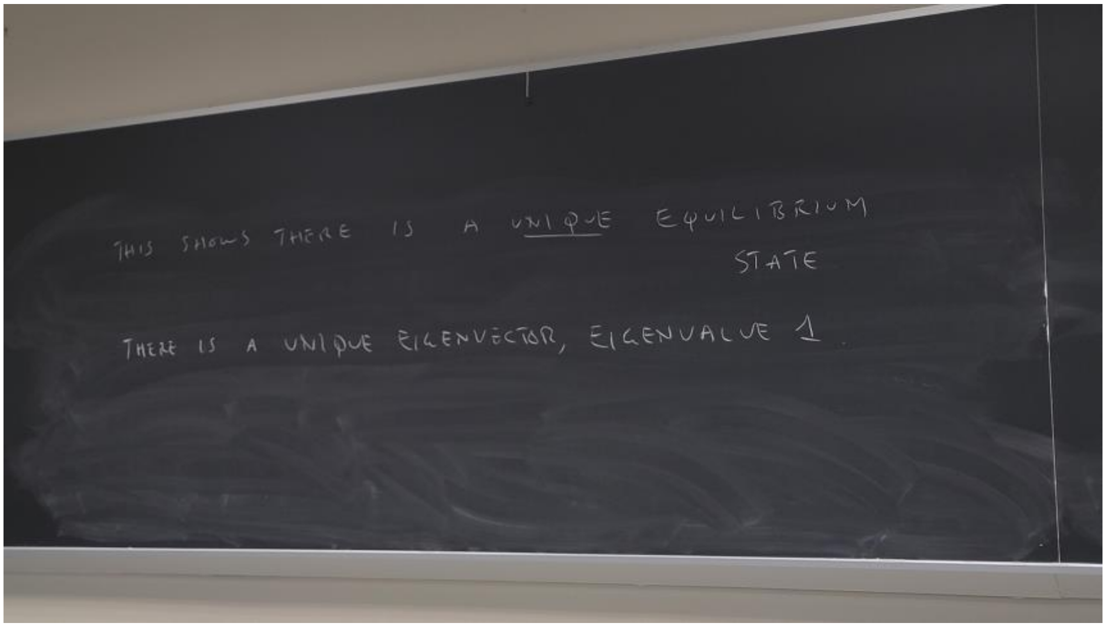
$$\underline{x} \cdot A = \begin{pmatrix} p_{11}x_1 + p_{21}x_2 + \ldots\ p_{n1}x_n \\ p_{12}x_1 + p_{22}x_2 + \ldots + p_{n2}x_n \\ \vdots \\ p_{1n}x_1 + \ldots + p_{nn}x_n \end{pmatrix}$$

$$\begin{aligned} \|\underline{x} \cdot A\| &= |p_{11}x_1 + \ldots + p_{n1}x_n| + \ldots + |p_{1n}x_1 + \ldots + p_{nn}x_n| \\ &\leq p_{11}|x_1| + \ldots + p_{n1}|x_n| + \ldots + |p_{1n}|x_1| + \ldots + p_{nn}|x_n|. \end{aligned}$$

$p_{11} + p_{21} + \ldots + p_{n1} = 1$ $= |x_1| + \ldots + |x_n| \Rightarrow |\lambda| \leq 1.$


NOTICE IF $\underline{x} = x_1 + \dots + x_n$ HAS NON-NEGATIVE ENTRIES.
$x_1 + \dots + x_n$ TOTAL WEIGHT.
$\underline{x}A$ SAME TOTAL WEIGHT, ENTRIES STILL NON-NEGATIVE.
$\underline{x}, \underline{x}A, \underline{x}A^2, \underline{x}A^3, \dots$
$\underline{x}_0, \underline{x}_1, \underline{x}_2, \dots$ $\underline{x}_n = \underline{x}A^n$.
THIS SEQUENCE OCCURS IN A CLOSED BOUNDED SPACE, SO USING REAL ANALYSIS YOU CAN FIND A SUBSEQUENCE
$\underline{x}_{n_1}, \underline{x}_{n_2}, \underline{x}_{n_3}, \dots$ WHICH CONVERGES TO A LIMIT $\underline{x}^*$.


x^* IS THE EQUILIBRIUM STATE WHICH WE SEEK.
OBSERVATION: IF $\underline{y}$ IS A VECTOR OF TOTAL MASS 0
THEN $\underline{y}A^n \to \underline{0}$ AS $n \to \infty$.
IF ASSUME THIS, $\underline{x} - \underline{x}^*$ IF $\underline{x}$ POSITIVE, SAME TOTAL MASS,
$(\underline{x} - \underline{x}^*)A^n \to 0$ AS $n \to \infty$. THIS CAN CHECK ANY TWO SUBSEQUENTIAL LIMITS EQUAL.


THIS SHOWS THERE IS A UNIQUE EQUILIBRIUM STATE
THERE IS A UNIQUE EIGENVECTOR, EIGENVALUE 1