Homework 2 solutions

Aleksander Doan

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Problem 1. First we show that $a^{-1} > 0$. Indeed, suppose by contradiction that $a^{-1} \leq 0$. Since $a \neq 0$, that would imply $a^{-1} < 0$. Then after multiplying both sides by a we would have $1 = aa^{-1} < 0$ which is a contradiction. (In any ordered field we must have $1 > 0$: prove it!) This shows that $a^{-1} > 0$ and likewise $b^{-1} > 0$, so $a^{-1}b^{-1} > 0$. Multiplying the inequality

$$0 < a < b$$

by $a^{-1}b^{-1}$ we obtain

$$0 < b^{-1} < a^{-1}$$

as desired.

Problem 2. If n is the product of primes, then so is $-n$, so it is enough to prove the statement for positive integers. For $n > 1$ let $P(n)$ be the statement: n is either prime or the product of primes. We prove the statement by (generalised) induction. Clearly $P(2)$ is true because 2 is prime. Suppose now that $P(k)$ holds for all $2 \leq k \leq n$. We want to conclude that $P(n+1)$ holds. If $n+1$ is prime, then the statement holds. On the other hand, if $n+1$ is not prime, then by definition there are divisors c,d such that $1 < c < n+1$, $1 < d < n+1$ and $n+1 = cd$. By the induction hypothesis, both c and d are the products of primes. Therefore, $n+1 = cd$ is also the product of primes, which proves that $P(n+1)$ holds. By the induction principle, $P(n)$ holds for all n.

Problem 3. Let $x \in A$ and $y \in B$. Then

$$x + y \leq \sup(A) + \sup(B).$$

Since the inequality holds for all such x and y it follows by the definition of $\sup(A+B)$ as the lowest upper bound that

$$\sup(A+B) \leq \sup(A) + \sup(B).$$

To show that $\sup(A+B) = \sup(A) + \sup(B)$ it remains to prove the reverse inequality. Fix $x \in A$. Since $\sup(A+B)$ is an upper bound for elements of $A+B$, for any $y \in B$ we have

$$x + y \leq \sup(A+B),$$

or equivalently

$$y \leq \sup(A+B) - x.$$

This inequality holds for all $y \in B$ so we conclude that

$$\sup(B) \leq \sup(A+B) - x,$$

or equivalently

$$x \leq \sup(A+B) - \sup(B).$$

But $x \in A$ was chosen arbitrarily so this inequality is true for all such x. Therefore,

$$\sup(A) \leq \sup(A+B) - \sup(B),$$

or equivalently

$$\sup(A) + \sup(B) \leq \sup(A+B),$$

which finishes the proof.

Problem 4. First we prove that $f(0_1) = 0_2$. We have

$$f(0_1) + f(0_1) = f(0_1 + 0_1) = f(0_1)$$

so after subtracting $f(0_1)$ from both sides we get $f(0_1) = 0_2$. Now we prove that $f(1_1) = 1_2$. First of all, observe that $f(1_1) \neq 0_2$. Indeed, we already know that $f(0_1) = 0_2$ and f is injective. Since $f(1_1) \neq 0_2$, there exists an inverse $f(1_1)^{-1}$. Consider the equality

$$f(1_1)f(1_1) = f(1_1 \cdot 1_1) = f(1_1).$$

Multiplying both sides by $f(1_1)^{-1}$ we get $f(1_1) = 1_2$ as desired. Next, we prove the remaining properties of f. For any $x \in F_1$ we have

$$f(-x) + f(x) = f(x - x) = f(0_1) = 0_2$$

so after subtracting $f(x)$ from both sides we get $f(-x) = -f(x)$. If we also have $x \neq 0_1$ then $f(x) \neq 0_2$ since, as before, $f(0_1) = 0_2$ and f is injective. Then

$$f(x^{-1})f(x) = f(x^{-1}x) = f(1_1) = 1_2$$

and multiplying both sides by $f(x)^{-1}$ we get $f(x^{-1}) = f(x)^{-1}$.

Problem 5. We prove the formula by induction on n. For $n = 1$ we have

$$\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = \binom{1}{0} a + \binom{1}{1} b = a + b,$$

so the statement is true. Assume that the statement holds for some $n \geq 1$. In order to prove it for $n + 1$ we compute

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k}
\end{aligned}$$

where in the last passage we just changed numbering by replacing k by $k+1$. Now split the term $k = n + 1$ from the first sum and the term $k = 0$ from the second sum to obtain

$$\begin{aligned}
(a+b)^{n+1} &= \binom{n}{n} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1} \\
&= a^0 b^{n+1} + \sum_{k=1}^{n} \left\{ \binom{n}{k-1} a^k b^{n+1-k} + \binom{n}{k} a^k b^{n+1-k} \right\} + a^{n+1} b^0.
\end{aligned}$$

Using the identity from the hint, we arrive at

$$\begin{aligned}
(a+b)^{n+1} &= a^0 b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + a^{n+1} b^0 \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k},
\end{aligned}$$

which proves that the statement is true for $n+1$. By the induction principle, it holds for all $n \geq 1$.

Bonus problem. Multiplying both sides by b, the claimed inequality may be written in the equivalent form

$$\left| b\sqrt{2} - a \right| \geq \frac{1}{2b\sqrt{2} + 1}. \tag{1}$$

Note that (1) is trivial if $a > \lceil b\sqrt{2} \rceil$, since in this case, the LHS is greater than 1, while the RHS is less than 1, so assume that $a \le \lceil b\sqrt{2} \rceil \le b\sqrt{2} + 1$. Now

$$\left|2b^2 - a^2\right| = \left|b\sqrt{2} + a\right| \left|b\sqrt{2} - a\right|.$$

The left hand side is an integer and is not 0, since $\sqrt{2} \notin \mathbb{Q}$, hence is at least 1. It follows that

$$\left|b\sqrt{2} - a\right| \ge \frac{1}{b\sqrt{2} + a} \ge \frac{1}{2b\sqrt{2} + 1},$$

as desired.