Homework 13 solutions

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Problem 1. Let n be a positive integer satisfying $x \in [n\pi, (n+1)\pi)$. Write

$$\int_0^x \frac{\sin t}{1+t} dt = \sum_{k=0}^{n-1} \int_k^{k+1} \frac{\sin t}{1+t} dt + \int_n^x \frac{\sin t}{1+t} dt.$$

Observe that sin is positive on the intervals of the form $(2k\pi, (2k+1)\pi)$ and negative on the intervals of the form $((2k-1)\pi, 2k\pi)$. In the sum above, consider the sum of two consecutive terms:

$$\int_{2k\pi}^{(2k+1)\pi} \frac{\sin t}{1+t} dt + \int_{(2k+1)\pi}^{(2k+2)\pi} \frac{\sin t}{1+t} dt.$$

Integrating by substitution, we obtain the sum

$$\begin{aligned} &\int_0^\pi \left\{ \frac{\sin(u + 2k\pi)}{1 + 2k\pi + u} + \frac{\sin(u + 2(k+1)\pi)}{1 + (2k+1)\pi + u} \right\} du \\ &= \int_0^\pi \sin u \left\{ \frac{1}{1 + 2k\pi + u} - \frac{1}{1 + (2k+1)\pi + u} \right\} du > 0, \end{aligned}$$

where we have used that $\sin(u + 2k\pi) = \sin u$ and $\sin(u + (2k+1)\pi) = -\sin u$. This shows that the sum of every consecutive terms is positive and so the integral is positive. We also have to deal with the last term (the integral from n to x). If n is even, then this integral is positive. If n is odd, then this integral can be paired with the previous term which is positive and the calculation above shows that their sum is positive.

Problem 2. We have

$$e^x = P_n(x) + Q_n(x)$$

where

$$Q_n(x) = \sum_{k=2n+1}^{\infty} \frac{x^k}{k!}.$$

Note that this power series is absolutely convergent on $(-\infty, \infty)$. Observe that P_n is an even degree polynomial and so $\lim_{x\to\pm\infty} P_n(x) = \infty$. Since P_n is a continuous function, this shows that P_n must achieve its global minimum at some point, x_0 say. At x_0 we have

$$0 = P_n'(x_0) = e^{x_0} - Q_n'(x_0) = e^{x_0} - \sum_{k=(2n+1)}^{\infty} k \frac{x_0^{k-1}}{k!} = e^{x_0} - \sum_{k=2n}^{\infty} \frac{x_0^k}{k!}$$

or equivalently

$$e^{x_0} = \sum_{k=2n}^{\infty} \frac{x_0^k}{k!}.$$

Using this relation, we compute the value of P_n at x_0:

$$P_n(x_0) = e^{x_0} - Q_n(x_0) = \sum_{k=2n}^{\infty} \frac{x_0^k}{k!} - \sum_{k=2n+1}^{\infty} \frac{x_0^k}{k!} = \frac{x_0^{2n}}{(2n)!}.$$

In particular, we see that $x_0 \neq 0$ because $P_n(0) = 1$. Thus, the right-hand side is strictly positive. Since the value of P_n at its global minimum is positive, we see that $P_n(x) > 0$ for all $x \in (-\infty, \infty)$.

Problem 3. Let us discuss only the third example as the other ones are similar. Suppose that we have a solution of the form $y = \sum_k a_k x^k$, where the series converges absolutely. Then

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1},$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

The equation $y'' + xy' + y = 0$ is equivalent to

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

or after renumbering,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

Comparing the coefficients next to the powers of x, we obtain the relation $2a_2 + a_0 = 0$ and for $k = 1, 2, \ldots$

$$(k+2) a_{k+2} + a_k = 0.$$

This shows that a_0 and a_1 can be chosen arbitrarily and the remaining coefficients are

$$a_{2k} = (-1)^k \frac{a_0}{2 \cdot 4 \cdots (2k)},$$

$$a_{2k+1} = (-1)^k \frac{a_1}{3 \cdot 5 \cdots (2k+1)}.$$

Using the ratio test we easily see that for any choice of a_0 and a_1 the radius of convergence of such a power series is infinite and so $y = \sum_k a_k x^k$ indeed defines an analytic function solving the given differential equation.

Problem 4. Let s be the square signal from Lecture 23, page 37. We know that the Fourier series of s is

$$s(x) \sim \frac{-2i}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{2\pi i n x}$$

and Parseval's identity gives us $\zeta(2)$. By the convolution formula, the Fourier coefficients of the convolution $s * s$ are squares of the Fourier coefficients of s:

$$s * s \sim -\frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{2\pi i n x}.$$

Applying Parseval's identity,

$$\int_0^1 (s*s)^2(x)dx = \frac{16}{\pi^4} \sum_{n \text{ odd}} \frac{1}{n^4} = \frac{32}{\pi^4} \sum_{n>0 \text{ odd}} \frac{1}{n^4}.$$

As in the proof from Lecture 23, we have

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n>0 \text{ odd}} \frac{1}{n^4}\left(1 + \frac{1}{2^4} + \frac{1}{2^8} + \cdots\right) = \sum_{n>0 \text{ odd}} \frac{1}{n^4}\left(\frac{1}{1-2^{-4}}\right) = \frac{16}{15} \sum_{n>0 \text{ odd}} \frac{1}{n^4}.$$

In order to determine $\zeta(4)$ it remains to find the integral of $(s*s)^2$. We have

$$\begin{aligned}(s*s)(x) &= \int_0^1 s(t)s(x-t)dt = \int_0^{1/2} s(t)s(x-t)dt + \int_{1/2}^1 s(t)s(x-t)dt \\ &= \int_0^{1/2} s(x-t)dt - \int_{1/2}^1 s(x-t)dt = \int_{x-1/2}^{x} s(u)du - \int_x^{x+1/2} s(u)du \\ &= 2\int_{x-1/2}^{x} s(u)du.\end{aligned}$$

where we have used that s is 1-periodic and the integral of s over any interval of length 1 is zero. We easily compute the last integral and obtain:

$$(s*s)(x) = \begin{cases} 4x-1 & \text{for } x \in [0,1/2), \\ -4x+3 & \text{for } x \in [1/2,1]. \end{cases}$$

Thus,

$$\int_0^1 (s*s)^2(x)dx = \int_0^{1/2} (4x-1)^2 dx + \int_{1/2}^1 (-4x+3)^2 dx = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

We conclude that

$$\frac{1}{3} = \int_0^1 (s*s)^2(x)dx = \frac{32}{\pi^4} \sum_{n>0 \text{ odd}} \frac{1}{n^4} = \frac{15}{16} \cdot \frac{32}{\pi^4} \zeta(4) = \frac{30}{\pi^4} \zeta(4)$$

and as a consequence $\zeta(4) = \pi^4/90$.

Problem 5. You can find many different proofs in this very nice Wikipedia article:

`https://en.wikipedia.org/wiki/Divergence_of_the_sum_of_the_reciprocals_of_the_primes`