Homework 12 solutions

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Problem 1. Set $b_n = a_{n+1} - a_n$ for $n = 1, \ldots$. Then $b_1 = a_2 - a_1$ and

$$2b_n = 2a_{n+1} - 2a_n = a_n + a_{n-1} - 2a_n = -a_n + a_{n-1} = -b_{n-1}.$$

By induction we prove that $b_n = (-1/2)^{n-1}(a_2 - a_1)$. On the other hand,

$$a_{n+1} - a_1 = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n+1} - a_n) = \sum_{k=1}^{n} b_k = (a_2 - a_1)\sum_{k=0}^{n-1}(-1/2)^k.$$

Using the formula for geometric series, we conclude that $\lim_{n\to\infty} a_n = g$ exists and equals

$$g = a_1 + \frac{2(a_2 - a_1)}{3}.$$

Problem 2. (1) If f has a local extremum at $c \neq 0$ then $f'(c) = 0$ and $f''(c) = c^{-1}(1 - e^{-c})$. We easily see that this expression is always positive no matter what the sign of c is. Therefore, c is a local minimum.

(2). Consider the function $g(x) = (1 - e^{-x})/x = \frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$. We easily check that g is continuous and $g(0) = 1$. Moreover, we have $f'' = g - 3(f')^2$, so in particular f'' is continuous and, if 0 is a local extremum, $f''(0) = 1 > 0$. We conclude that in this case 0 is a local minimum.

(3). For $x > 0$ we have

$$f''(x) \leq f''(x) + (f'(x))^2 = \frac{1 - e^{-x}}{x} < 1.$$

Integrating twice from 0 to x and using $f(0) = f'(0)$, we get

$$f(x) \leq \frac{x^2}{2}$$

for $x \geq 0$. To see that this is an optimal constant, observe that

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + o(x^2) = \frac{1}{2}x^2 + o(x^2),$$

where we have used that $f''(0) = 1$ as proved earlier. If we have $f(x) \leq Ax^2$ for some A,

$$\frac{1}{2} = \lim_{x\to 0^+} \frac{f(x)}{x^2} \leq \lim_{x\to 0^+} \frac{Ax^2}{x^2} = A.$$

So $A \geq 1/2$ and $1/2$ is the minimal such a constant.

Problem 3. (1) Since $\lim_{x\to 0^+} x^{1/4}|\log x| = 0$ (which we prove, for example, using l'Hôpital's rule), there exists a positive constant C such that $|\log x| < Cx^{-1/4}$ for all $x \in (0,1]$. Thus,

$$\int_0^1 \frac{|\log x|}{\sqrt{x}} dx < C\int_0^1 x^{-1/2-1/4}dx = C\int_0^1 x^{-3/4}dx = 4C[x^{1/4}]_0^1 = 4C$$

and the integral converges absolutely by the comparison theorem.

(2) The integral converges. Around zero the integral can be compared to the integral of $\log x$ from 0 to some $\epsilon > 0$. This integral converges since $\int \log x dx = x(\log x - 1) + C$ has a finite limit as $x \to 0^+$. On the other hand, the integrand $\log x/(1-x)$ converges to one as $x \to 1$ (which we check using l'Hôpital's rule), so the integral over a neighbourhood of 1 also converges.

(3) The integral diverges. Observe that after a change of variables, the absolute value of the integral of $(\sqrt{x}\log x)^{-1}$ over a neighbourhood of 1 can be estimated below by

$$\int_0^\epsilon \frac{dx}{|\log(1-x)|}$$

for some $\epsilon > 0$. (Up to a constant: we use here that $x^{-1/2}$ is bounded below when $x \in [\frac{1}{2}, 1]$, say.) The last integral diverges. To see this, observe that $\lim_{x\to 0^+} x^{-1}|\log(1-x)| = 1$, so there is a constant C such that $|\log(1-x)| \le Cx$ for $x \in (0,\epsilon]$. Then

$$\int_0^\epsilon \frac{dx}{|\log(1-x)|} \ge C\int_0^\epsilon \frac{dx}{x} = \infty,$$

which proves the divergence of the integral.

(4) Use the substitution $u = \log x$, $du = 1/xdx$:

$$\int_2^\infty \frac{dx}{x(\log x)^3} = \int_{\log 2}^\infty u^{-3}du = -\frac{1}{2}[u^{-2}]_{\log 2}^\infty = \frac{1}{2(\log 2)^2}.$$

so the integral converges.

Problem 4. We compute

$$\int \left(\frac{x}{2x^2+2C} - \frac{C}{x+1}\right) dx = \frac{1}{4}\log(C+x^2) - C\log(x+1) + const = \frac{1}{4}\log\left(\frac{C+x^2}{(x+1)^{4C}}\right) + const$$

The expression

$$\frac{C+x^2}{(x+1)^{4C}}$$

has a finite limit as $x \to \infty$ if and only if $C = 1/2$, in which case the limit is equal to 1. Thus, the integral in question is equal to

$$\lim_{N\to\infty}\left[\frac{1}{4}\log\left(\frac{1/2+x^2}{(x+1)^2}\right)\right]_1^N = -\frac{1}{4}\log\left(\frac{3}{8}\right).$$

Problem 5. We use the formula for the radius of convergence of $\sum_n a_n x^n$:

$$r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

For example, in (1) we have

$$r = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n^3}} = 1,$$

where we have used that

$$\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} \exp(\log(n^{3/n})) = \exp\left(\lim_{n\to\infty} \frac{3\log n}{n}\right) = e^0 = 1.$$

(The limit $\log n/n$ can be computed using for example l'Hôpital's rule.) To express series (1) in terms of elementary functions, we consider

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Then

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1},$$

$$xf'(x) = \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n,$$

and we continue (the derivatives below can be easily computed...)

$$(xf'(x))' = \sum_{n=1}^{\infty} n^2 x^{n-1},$$

$$x(xf'(x))' = \sum_{n=1}^{\infty} n^2 x^n,$$

$$(x(xf'(x))')' = \sum_{n=1}^{\infty} n^3 x^{n-1},$$

$$x(x(xf'(x))')' = \sum_{n=1}^{\infty} n^3 x^n.$$

Problem 6. By the Cauchy-Schwarz inequality,

$$\sum_n \frac{\sqrt{a_n}}{n} \le \sqrt{\left(\sum_n a_n\right)\left(\sum_n \frac{1}{n^2}\right)} < \infty.$$