Homework 11 solutions

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Problem 1. (1) and (3) are telescoping series, (2) and (4) are geometric series. To prove that (4) converges we use the identity

$$\frac{\sqrt{a}-\sqrt{b}}{a-b} = \frac{1}{\sqrt{a}+\sqrt{b}},$$

so that the n-th term of the series (4) is

$$\frac{1}{(\sqrt{n+1}+\sqrt{n})\sqrt{n^2+n}}.$$

The denominator satisfies the inequality

$$(\sqrt{n+1}+\sqrt{n})\sqrt{n^2+n} > n^{3/2},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n+1}+\sqrt{n})\sqrt{n^2+n}} < \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty,$$

where we have used the comparison test and the fact that $\sum_n 1/n^s$ converges if and only if $s > 1$ (which was proved in class using the integral test).

Problem 2. Let a_n be the n-th element of the sequence. We have $a_1 = 1$ and

$$a_{n+1} = \sqrt{1+a_n}. \tag{1}$$

Clearly all a_n are positive. Suppose that the limit $a = \lim_{n\to\infty} a_n$ exists. Then passing to the limit $n \to \infty$ in (1) and using the continuity of $x \mapsto \sqrt{1+x}$ we get

$$a = \sqrt{1+a}$$

or equivalently

$$a^2 - a - 1 = 0,$$

from which it follows that $a = (1+\sqrt{5})/2$ (the other root is negative). It remains to show that a_n indeed converges. We do it by proving that it is bounded above and increasing.

First, we show by induction by it is bounded above by a. Clearly $a_1 = 1 < a$. Assume that $a_n < a$. Then

$$a_{n+1}^2 = 1 + a_n < 1 + a = a^2$$

so $a_{n+1} < a^2$ (since both numbers are positive) and by induction the statement is true for all n. Now we show that the sequence is increasing. This is equivalent to showing that

$$0 < a_{n+1}^2 - a_n^2 = 1 + a_n - a_n^2.$$

The function $x \mapsto 1 + x - x^2$ has root a and is positive on the interval $[0, a]$. Since we have already shown that $a_n \in [0, a]$, it follows that the inequality above is satisfied and the sequence a_n is increasing. This finishes the proof.

Problem 3. Since the function $x \mapsto 1/x$ is decreasing on $(0, \infty)$ we have

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > \int_1^n \frac{dx}{x} = \log n.$$

(Draw the graph of $x \mapsto 1/x$ to see this. Compare also with the proof of the integral test from Lecture 19.) As a consequence, $a_n > 0$ for all n, so the sequence in question is bounded below. We will show now that it is also decreasing and then it follows that it converges. Consider the difference of two consecutive terms

$$a_{n+1} - a_n = \frac{1}{n+1} - \log(n+1) + \log(n) = \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) = f(n),$$

where

$$f(x) = \frac{1}{x+1} - \log\left(1 + \frac{1}{x}\right).$$

We claim that $f(x) < 0$ for all x positive. This implies that that $a_{n+1} < a_n$ and the sequence is decreasing. The claim is easy to establish. First, a quick calculation shows that $f'(x) > 0$ for x positive, so f is increasing. On the other hand, $\lim_{x\to\infty} f(x) = 0$, so we conclude that $f(x) < 0$ for all x positive, as desired.

Problem 4. The sequence can be defined inductively by $a_1 = 1/2$ and $a_{n+1} = 1/(2+a_n)$. Note that $a_n > 0$ for all n (easy induction). If the limit $a = \lim_{n\to\infty} a_n$ exists, then passing to $n \to \infty$ in the above recurrence relation (as we did in Problem 2) gives

$$a = \frac{1}{2+a},$$

which has a unique positive solution $a = \sqrt{2} - 1$. To justify passing to the limit, we show in the same way as in Problem 2 that a_n is bounded below by a and decreasing.

Problem 5. The sum of the numbers in every column is the same and equal to zero because

$$1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = 0.$$

Therefore, if we first sum the columns and then the rows, we obtain the sum of zeroes, which is zero. On the other hand, the sum of the numbers in the i-th row is equal to $-1/2^{i-1}$. Summing all these sums, we get

$$-1 - \frac{1}{2} - \frac{1}{4} - \cdots = -2.$$

Problem 6. Because of the periodicity of sin we can write the integral as an alternating series

$$\int_{\pi/2}^{\infty} \frac{\sin(x)}{\sqrt{x}} dx = \sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

where

$$a_n = \int_{n\pi/2}^{(n+1)\pi/2} \frac{|\sin x|}{x} dx = \int_0^{\pi/2} \frac{|\sin x|}{\sqrt{x+n}} dx.$$

One easily checks that the sequence a_n is decreasing and converges to zero. Thus, by the alternating series test, the sum $\sum_n (-1)^{n+1} a_n$ converges.