Homework 10 solutions

Aleksander Doan

November 21, 2016

Problem 1. We use the formula from Lecture 16 to find the general form of a solution:

$$y(x) = \frac{C + x}{\sin x}$$

Since $x/\sin x \to 1$ as $x \to 0$, we see that the only solution having a finite limit as $x \to 0$ is the one with $C = 0$. Likewise, the only possible solution with a finite limit as $x \to \pi$ is the one with $C = -\pi$ (since otherwise the denominator converges to zero and the numerator converges to a non-zero number $C + \pi$).

Problem 2. Again, we use the formula from Lecture 16 to solve the equation:

$$y(x) = \frac{C}{\sin x} - \frac{1}{2}\frac{\cos(2x)}{\sin x} = \frac{C - 1/2}{\sin x} + \sin x.$$

The only solution that extends to $(-\infty, \infty)$ is the one with $C = 1/2$.

Problem 3. Let $f(t)$ be the temperature of the thermometer at time t, measure from the moment $t = 0$ when the temperature is $75F$. We have $f(0) = b = 75$ and $f(5) = 65$, $f(10) = 60$. The temperature M of the environment is constant. From Lecture 16 we know that $f(t)$ is given by the formula

$$f(t) = be^{-kt} + Me^{-kt}\int_0^t ke^{ku}du = be^{-kt} + Me^{-kt}(e^{kt} - 1) = (b - M)e^{-kt} + M.$$

Setting $t = 5$ and $t = 10$ gives us the equations

$$65 - M = (75 - M)e^{-5k},$$

$$60 - M = (75 - M)e^{-10k}.$$

Squaring the first equation and dividing it by the second equation, we obtain

$$\frac{(65 - M)^2}{60 - M} = 75 - M,$$

or equivalently

$$(65 - M)^2 = (60 - M)(75 - M),$$

which has a unique solution $M = 55$.

Problem 4. Suppose for simplicity that $\omega = L = R = 1$. Then, as we learnt in Lecture 16, the solution is given by

$$f(t) = e^{-t}\int_0^t \sin x e^x dx = \frac{1}{2}e^{-t}\left[e^x(\sin x - \cos x)\right]_0^t = \frac{1}{2}(\sin x - \cos x) + \frac{1}{2}e^{-t}.$$

To finish the proof, observe that any sum of the form

$$A\sin x + B\cos x$$

can be written as $C\sin(x+\alpha)$ for some C and α. Namely, take $C = \sqrt{A^2+B^2}$ and write

$$A\sin x + B\cos x = C\left(\frac{A}{C}\sin x + \frac{B}{C}\cos x\right).$$

Since A/C and B/C are between $[-1,1]$ and the sum of their square is equal to zero, there exists α such that $A/C = \cos\alpha$ and $B/C = \sin\alpha$ (prove it!). Then

$$A\sin x + B\cos x = C(\cos\alpha\sin x + \sin\alpha\cos x) = C\sin(x+\alpha).$$

Problem 5. Direct calculations using the last theorem of Lecture 16. First, solve the homogenous equation, then construct a particular solution using the Wronskian. The solutions are

1. $y(x) = C_1\sin x + C_2\cos x - \frac{1}{2}x\cos x$,
2. $y(x) = C_1e^{3x} + C_2 - \frac{3}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x$,
3. $y(x) = C_1\sin(2x) + C_2\cos(2x) + x\sin x - \frac{2}{3}\cos x$,
4. $y(x) = C_1e^{-2x} + C_2e^x + \frac{1}{3}e^x x + \frac{1}{4}e^{2x}$.

Problem 6. By the Fundamental Theorem of Calculus,

$$f(x) = f(a) + \int_a^x f'(t)dt = \int_a^x f'(t)dt.$$

Applying the integral Cauchy-Schwarz inequality to the functions 1 and $f'(t)$, we obtain

$$\begin{aligned}|f(x)| = \left|\int_a^x f'(t)dt\right| \le \int_a^x |f'(t)|dt &\le \sqrt{\int_a^x 1dt\int_a^x (f'(t))^2dt} \\ \le \sqrt{\int_a^b 1dt\int_a^b (f'(t))^2dt} &\le \sqrt{b-a}\sqrt{\int_a^b (f'(t))^2dt},\end{aligned}$$

where we have used that both 1 and $(f')^2$ are non-negative so by enlarging the interval of integration we increase the integral. Squaring both sides and integrating from a to b we obtain

$$\int_a^b (f(x))^2dx \le \int_a^b \left\{(b-a)\int_a^b (f'(t))^2dt\right\}dx = (b-a)^2\int_a^b (f'(t))^2dt,$$

since the expression in the curly bracket does not depend on x, and so we integrate a constant function from a to b, as a result obtaining the extra factor $(b-a)$.