MATH 141, FALL 2016, HW9

DUE IN SECTION, NOVEMBER 1

Problem 1. Let $f(x)$ be a function defined on $[0, \infty)$ which is increasing and bounded above. Prove that

$$\lim_{x\to\infty} f(x) = \sup\{f(x) : x \in [0, \infty)\}.$$

Problem 2. Evaluate

$$\lim_{n\to\infty} 4^n \left(1 - \cos\frac{\theta}{2^n}\right).$$

Problem 3. Find the polynomial $P(x)$ of lowest degree such that $\sin(x - x^2) = P(x) + o(x^6)$ as $x \to 0$.

Problem 4. Evaluate the following limits.

$$\lim_{x\to 0} \frac{\log(1+x)}{e^{2x} - 1}, \qquad \lim_{x\to 0}(x + e^{2x})^{1/x}.$$

Problem 5. Find the following limits:

(1) $\lim_{x\to 0} \frac{e-(1+x)^{\frac{1}{x}}}{x}$.
(2) $\lim_{n\to\infty} \frac{n}{\log n}[n^{\frac{1}{n}} - 1]$.
(3) $\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)}$.
(4) $\lim_{x\to 0} \frac{x - \sin x}{\tan x - x}$.

Problem 6. For any real number $\lambda \geq 1$, denote by $f(\lambda)$ the real solution to the equation $x(1 + \ln x) = \lambda$. Prove that

$$\lim_{\lambda\to\infty} \frac{f(\lambda)}{\frac{\lambda}{\log\lambda}} = 1.$$

Bonus Problem. Prove that $2^{2n}\sqrt{n}\int_0^1 x^n(1-x)^n dx$ converges to a limit as $n \to \infty$. Evaluate the limit.