MATH 141, FALL 2016, HW8

DUE IN SECTION, OCTOBER 25

Problem 1. (27 pts) Perform the following integrals

(1) $\int \frac{dx}{x(x^2+1)^2}$
(2) $\int (2+3x)\sin 5x dx$
(3) $\int_1^2 x^{-2} \sin \frac{1}{x} dx$
(4) $\int \sqrt{1+3\cos^2 x} \sin 2x dx$
(5) $\int x^2 \sin^2 x dx$
(6) $\int_0^1 x^4(1-x)^{20} dx$
(7) $\int \frac{dx}{x^3-x}$
(8) $\int \frac{x}{\sqrt{x^2+x+1}} dx$.
(9) For natural numbers n, $\int_0^\infty x^n e^{-x} dx$ (the improper integral $\int_0^\infty$ means take $\int_0^T$ and let the upper limit $T \to \infty$).

Problem 2. Show that for $x > 0$, $\frac{x}{1+x} < \log(1+x) < x$ and for all $x \neq 0$, $e^x > 1+x$.

Problem 3. Evaluate $\lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$.

Problem 4. Prove that if m and n are integers, then

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0.$$

Problem 5. (Riemann-Lebesgue lemma) Let f be integrable on $[a, b]$. Prove that

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x dx = 0.$$

(Hint: approximate f with a step function.)

Problem 6. Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous function, that is, for each $z \in \mathbb{C}$ and each $\epsilon > 0$ there is a $\delta > 0$ such that if w is a complex number and $|w - z| < \delta$, then $|f(w) - f(z)| < \epsilon$. Let $R > 0$. Prove that f is bounded below on $B_R(0) = \{z \in \mathbb{C} : |z| \leq R\}$, and achieves its minimum on this set, by completing the following steps.

(1) For real x, $|x| \leq R$, show that $f(x + iy)$ is continuous as a function of the real variable y, and hence $f(x + iy)$ achieves its minimum on the set $\{y : |x + iy| \leq R\}$.
(2) Define $F(x) = \min\{f(x + iy) : |x + iy| \leq R\}$. Show that $F(x)$ is continuous on $[-R, R]$, and hence achieves its minimum on this set. Check that this minimum is the minimum of f on $B_R(0)$.

Bonus Problem. Let f be an increasing function on $[0, 2\pi]$. Prove the bound, for each $n \geq 1$,

$$\left| \int_0^{2\pi} f(x) \cos nx dx \right| \leq \frac{f(2\pi) - f(0)}{n}.$$

Bonus Problem. Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(1) If $u \geq 0$ and $v \geq 0$, then
$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$
Equality holds if and only if $u^p = v^q$.
(2) If $f \geq 0$ and $g \geq 0$ are integrable on $[a, b]$, and
$$\int_a^b f^p = 1 = \int_a^b g^q,$$

then fg is integrable and

$$\int_a^b fg \leq 1.$$

(3) *Hölder's inequality*: If f and g are integrable on $[a,b]$, then

$$\left|\int_a^b fg\right| \leq \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left(\int_a^b |g|^q\right)^{\frac{1}{q}}.$$