MATH 141, FALL 2016, HW6

DUE IN SECTION, OCTOBER 11

Problem 1. Let m, n be positive integers. Compute

$$\lim_{x\to 1} \frac{1-x^m}{1-x^n}.$$

Problem 2. Prove that if f is integrable on $[a, b]$, then for any $\epsilon > 0$ there are continuous functions $g \leq f \leq h$ with $\int_a^b h(x)dx - \int_a^b g(x)dx < \epsilon$.

Problem 3. Suppose that f is continuous and $\lim_{x\to\infty} f(x) = a$. Prove that

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = a.$$

Problem 4. A function f is convex on an interval if, for all x, y in the interval and all $0 < t < 1$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Prove that a convex function on $\mathbb{R}$ or on any open interval, must be continuous.

Problem 5. A homeomorphism is a continuous map with continuous inverse. Give a homeomorphism between the open interval $(0, 1)$ and $\mathbb{R}$. Does there exist a homeomorphism between $(0, 1)$ and $[0, 1]$?

Bonus Problem. Let $a_0, ..., a_n$ be real numbers satisfying $|a_0| + ... + |a_{n-1}| < a_n$. Prove that the trigonometric polynomial

$$a_0 + a_1 \cos x + ... + a_n \cos nx$$

has at least $2n$ zeros on $[0, 2\pi)$.

Bonus Problem. Given an example of a homeomorphism between the closed disc in $\mathbb{R}^2$, $B_2 = \{(x, y) : x^2 + y^2 \leq 1\}$ and the equilateral triangle T with vertices at $(1, 0)$, and $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, edges and interior included.

Bonus Problem. Show that the equations

$$y = \sin(\cos x), \qquad x = \cos(\sin y)$$

have a solution in $\mathbb{R}^2$. Do they have more than one? Answer the same questions for the equations

$$y = \sin\left(\frac{\pi}{2}\cos\frac{\pi}{2}x\right), \qquad x = \sin\left(\frac{\pi}{2}\cos\frac{\pi}{2}y\right).$$