MATH 141, FALL 2016, HW3

DUE IN SECTION, SEPTEMBER 15

Problem 1. If x is rational, $x \neq 0$ and y is irrational, prove that $x+y, x-y, xy, x/y$, and y/x are all irrational.

Problem 2. Let α and β be Dedekind cuts. Prove that $\alpha+\beta = \{x+y : x \in \alpha, y \in \beta\}$ is a Dedekind cut.

Problem 3. Suppose that $y - x > 1$. Prove that there is an integer n with $x < n < y$. Now show that if x and y are arbitrary real numbers with $x < y$, then there is a rational z with $x < z < y$. Finally, show that if $x < y$ are rational numbers there is an irrational number z with $x < z < y$.

Problem 4. Let $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$ denote the sequence of Fibonacci numbers. Prove that for all $n \geq 1$, $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$.

Problem 5. Let $S_1, S_2, S_3, ...$ be a sequence of sets, each of which is countable. Prove that $S = \bigcup_{n=1}^{\infty} S_n$ is countable. (Hint: find a map into $\mathbb{N}^2$.)

Problem 6. (Division algorithm for polynomials) Let $B(x)$ denote a non-zero polynomial with coefficients in a field $\mathbf{F}$. For each polynomial $P(x)$ with coefficients in $\mathbf{F}$ prove that there exist unique polynomials $Q(x)$ and $R(x)$, also with coefficients in $\mathbf{F}$, such that

$$P(x) = Q(x)B(x) + R(x)$$

and such that $\deg R(x) < \deg B(x)$. [Use the convention that 0 is a polynomial of degree less than 0.]

Bonus Problem. Show that there are integers n_1, n_2, n_3, not all zero, satisfying $\max(|n_1|, |n_2|, |n_3|) < 1000$, such that

$$\left| n_1 + n_2\sqrt{2} + n_3\sqrt{3} \right| < 4.2 \times 10^{-6}.$$