MATH 141, FALL 2016, HW2

DUE IN SECTION, SEPTEMBER 8

Problem 1. Prove from the axioms of an ordered field that if $0 < a < b$ then $0 < b^{-1} < a^{-1}$.

Problem 2. Let n and d denote integers. We say that d is a *divisor* of n if $n = cd$ for some integer c. An integer $n > 1$ is *prime* if the only positive divisors of n are 1 and n. Prove by induction that every integer $n > 1$ is either prime or the product of primes.

Problem 3. Let A and B be two non-empty sets of real numbers which are bounded above, and let $A+B$ denote the set of all numbers $x+y$ with $x \in A$ and $y \in B$. Prove that

$$\sup(A + B) = \sup(A) + \sup(B).$$

Problem 4. Let F_1, F_2 be fields, and let $f : F_1 \to F_2$ be an injective map satisfying $f(a+b) = f(a)+f(b)$ and $f(ab) = f(a)f(b)$. Indicate the identities in F_1 by $0_1, 1_1$ and in F_2 by $0_2, 1_2$. Prove $f(0_1) = 0_2, f(1_1) = 1_2$, and for all $x \in F_1 \setminus \{0\}$, $f(-x) = -f(x), f(x^{-1}) = f(x)^{-1}$.

Problem 5. Use induction to prove the *Binomial Theorem*: For all $n \in \mathbb{Z}_{>0}$, for all $a, b \in \mathbb{R}$,[1]

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Deduce that for all $n \geq 1$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

(Hint: first prove the identity, for $1 \leq k < n$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.)

Bonus Problem. Let a and b be positive integers. Show that $\left|\sqrt{2} - \frac{a}{b}\right| \geq \frac{1}{b(2b\sqrt{2}+1)}$.

[1] The factorial is defined by $0! = 1$, and for $n \geq 1$, $n! = n \cdot (n-1)!$. The binomial coefficient is, for $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.