MATH 141, FALL 2016, HW13

DUE IN SECTION, DECEMBER 8

Problem 1. Prove that for all $x > 0$,

$$\int_0^x \frac{\sin t}{1+t} dt > 0.$$

Prove that

$$\int_0^\infty \frac{\cos t}{1+t} dt > 0.$$

Problem 2. Show that, for all x and all natural numbers n,

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} > 0.$$

Problem 3. Assume that the following differential equations have a power series solution of the form $y = \sum a_n x^n$ and determine the nth coefficient a_n.

(1) $y' = \alpha y$
(2) $y'' = xy$
(3) $y'' + xy' + y = 0.$

Verify that the corresponding power series give genuine solutions of the differential equations.

Problem 4. Either by modifying the evaluation of $\zeta(2)$ from Lecture 23, or otherwise, prove that $\zeta(4) = \frac{\pi^4}{90}$.

Problem 5. Prove $\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty$.

Bonus Problem. Let $\mathcal{P}$ denote the set of prime numbers. Prove the logarithmic derivative identity in $\Re(s) > 1$,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{p \in \mathcal{P}} \frac{\log p}{p^s \left(1 - \frac{1}{p^s}\right)}.$$