Math 141: Lecture 9

Differentiation and it's properties

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Difference quotients and average velocity

Consider an object moving forward and backward along a number line.

- Let the particle's position at time t be $x(t)$
- The particle's average velocity between times t and $t+h$ is the *difference quotient*

$$\frac{\text{change in position}}{\text{change in time}} = \frac{x(t+h)-x(t)}{h}.$$

- Example: A ball thrown vertically in the air has position $x(t) = 28t - 4.9t^2$ meters for $0 \leq t \leq \frac{40}{7}$ seconds. It's average velocity between $t = 1$ and $t = 3$ is

$$\frac{39.9 - 23.1}{2} = 8.4 m/s$$

Instantaneous velocity

- If the limit exists, the *instantaneous velocity* of a particle at time t is the limit of difference quotients

$$v(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}.$$

- For instance, in the previous example where $x(t) = 28t - 4.9t^2$, when $0 \leq t + h \leq \frac{40}{7}$,

$$\frac{x(t+h) - x(t)}{h} = \frac{28h - 9.8th - 4.9h^2}{h} = 28 - 9.8t - 4.9h.$$

  Taking the limit as $h \to 0$,

$$v(t) = 28 - 9.8t.$$

Definition of derivative

Definition

The derivative $f'(x)$ is defined by the equation

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided the limit exists. The number $f'(x)$ is also called the *rate of change* of f at x.

Several other notations for derivatives are used, including $f'(x) = \frac{df}{dx}|_x = Df(x)$.

Definition of derivative

Definition

A function f is *differentiable on the right* at x if the right-hand limit

$$\lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{h}$$

exists. f is *differentiable on the left* at x if the left-hand limit

$$\lim_{h\uparrow 0} \frac{f(x+h)-f(x)}{h}$$

exists.

Definition of differentiable and continuously differentiable

Definition

- A function f on an interval I is *differentiable* on I if it is differentiable at each point of the interval, with the proviso that it is differentiable on the right (resp. on the left) at the left (resp. right) endpoint, if it is included in I.
- f is *continuously differentiable* at a point p if it is differentiable in a neighborhood of p, and if the derivative function is continuous at p.
- f is *continuously differentiable* on I if it is continuously differentiable at each point of I.

All of these notions have higher derivative notions, e.g. twice differentiable, twice continuously differentiable, ... A function which is n times differentiable for any n is said to be infinitely differentiable.

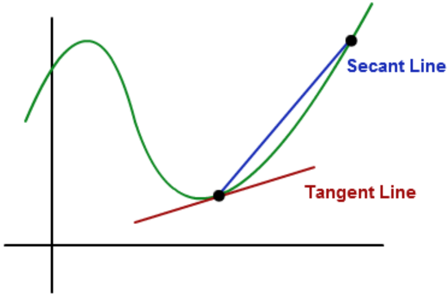
Geometric interpretation

Definition

Let f be differentiable at x. The *tangent line* to the graph of f at $(x, f(x))$ is the line passing through $(x, f(x))$ with slope $f'(x)$.

The difference quotients give the slopes of secant lines passing through $(x, f(x))$.

Geometric interpretation



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Acceleration and higher derivatives

- Just as velocity is the derivative of position, acceleration is the derivative of velocity. In the example given, the acceleration is

$$\lim_{h\to 0} \frac{(28 - 9.8(t+h)) - (28 - 9.8t)}{h} = -9.8 m/s^2$$

  This is the usual acceleration due to the Earth's gravity (near it's surface).

- The higher derivatives of a function, when they exist, are indicated

$$\begin{aligned} f^{(1)}(x) &= f'(x), \\ f^{(2)}(x) &= f''(x) = (f')'(x), \\ &\vdots \\ f^{(n)}(x) &= (f^{(n-1)})'(x). \end{aligned}$$

Derivative of a constant function

Let $f(x) = c$ for all x. The difference quotient is, for $h \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0.$$

Thus $f'(x) = 0$ for all x.

Derivative of a linear function

Let $f(x) = mx + b$ for all real x. The difference quotient is, for $h \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = \frac{m(x+h) + b - (mx+b)}{h} = m.$$

Thus $f'(x) = m$ for all x.

Derivative of an integer power function

Let $n \geq 1$ be an integer, and let $f(x) = x^n$. The difference quotient is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}.$$

Using the formula $a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$, one obtains

$$\frac{(x+h)^n - x^n}{h} = \sum_{k=0}^{n-1} (x+h)^k x^{n-1-k}.$$

By continuity,

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Derivative of the sine function

Recall the difference of sines formula,

$$\sin y - \sin x = 2 \sin \frac{y-x}{2} \cos \frac{y+x}{2}.$$

Thus, for $h \neq 0$,

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin(h/2)}{h/2} \cos\left(x + \frac{h}{2}\right).$$

Since $\lim_{h\to 0} \frac{\sin h/2}{h/2} = 1$, $\lim_{h\to 0} \cos\left(x + \frac{h}{2}\right) = \cos x$.
Thus

$$\sin' x = \cos x.$$

Derivative of the cosine function

Recall the difference of cosines formula,

$$\cos y - \cos x = -2 \sin \frac{y-x}{2} \sin \frac{y+x}{2}.$$

Thus, for $h \neq 0$,

$$\frac{\cos(x+h) - \cos x}{h} = -\frac{\sin(h/2)}{h/2} \sin\left(x + \frac{h}{2}\right).$$

Letting $h \to 0$, one obtains

$$\cos' x = -\sin x.$$

Derivative of the nth root function

Let $f(x) = x^{1/n}$ for $x > 0$. For $h \neq 0$ and $x + h > 0$, let $u = (x+h)^{1/n}$ and $v = x^{1/n}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{u - v}{u^n - v^n} = \frac{1}{u^{n-1} + u^{n-2}v + \ldots + v^{n-1}}.$$

By continuity of the function $x \mapsto x^{\frac{1}{n}}$, letting $h \to 0$ obtains

$$f'(x) = \frac{1}{nx^{1-\frac{1}{n}}}.$$

Continuity of functions having derivatives

Theorem

Let $f(x)$ be a function defined in a neighborhood of a point p, and suppose that $f'(p)$ exists. Then f is continuous at p.

Proof.

Write

$$f(p+h) = f(p) + h\left(\frac{f(p+h)-f(p)}{h}\right).$$

Letting $h \to 0$, the difference quotient tends to $f'(p)$. By the limit of products property,

$$\lim_{h\to 0} f(p+h) = f(p),$$

so f is continuous at p. $\square$

Continuous functions are not necessarily differentiable

Define $f(x) = |x|$. This function is continuous at $x = 0$. The difference quotients there are given by

$$\frac{f(h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1 & h > 0 \\ -1 & h < 0 \end{cases}.$$

Since the limits on the left and right do not agree, f is not differentiable at 0.

Algebraic properties of derivatives

Theorem

Let f and g be two functions defined on a common interval. At each point where f and g both have a derivative, their sum $f+g$, difference $f-g$ and product $f \cdot g$ all have derivatives. If $g \neq 0$ then the ratio f/g also has a derivative. The derivatives are given by the formulae

1. $(f+g)' = f' + g'$
2. $(f-g)' = f' - g'$
3. $(f \cdot g)' = f \cdot g' + g \cdot f'$
4. $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$ *at points x where $g(x) \neq 0$.*

Derivative of a sum

The difference quotient of a sum is given by

$$\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$
$$=\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}.$$

Taking the limit as $h \to 0$ gives $(f+g)'(x) = f'(x) + g'(x)$.

Derivative of a product

The difference quotient of a product is given by

$$\begin{aligned}&\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\&=f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}.\end{aligned}$$

Taking limits as $h \to 0$ gives

$$(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x).$$

Derivative of a reciprocal

The difference quotient for the reciprocal $\frac{1}{f}(x)$ is

$$\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \frac{f(x) - f(x+h)}{hf(x)f(x+h)}.$$

Taking limits as $h \to 0$ obtains

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f(x)^2}.$$

Combined with the product formula, this gives the formula for a quotient.

Derivative of a polynomial

If $f(x)$ is the polynomial $a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, combining the formula for the derivative of a power function, for a product, and for a sum obtains

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1.$$

Derivative of negative powers

From the reciprocal formula, one obtains a formula for the derivative of negative powers. For integer $m \geq 1$,

$$\left(\frac{1}{x^m}\right)' = -\frac{mx^{m-1}}{x^{2m}} = -mx^{-m-1}.$$

Derivative of rational powers

Let $x > 0$ and $f(x) = x^r$ with r rational. If $r = \frac{1}{n}$ we've already checked the formula

$$f'(x) = rx^{r-1}.$$

This formula may be checked for $r = \frac{m}{n}$ by applying the product/quotient formula and induction.

Derivatives of the other trig functions

Apply the reciprocal formula to $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$ to obtain

$$\sec' x = \frac{\sin x}{(\cos x)^2} = \sec x \tan x$$
$$\csc' x = \frac{-\cos x}{(\sin x)^2} = -\csc x \cot x.$$

By the quotient formula

$$\tan' x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$
$$\cot' x = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x.$$

The chain rule

Theorem

Let $f = u \circ v$ be the composition of two functions. Suppose that both derivatives $v'(x)$ and $u'(y)$ exist, where $y = v(x)$. Then $f'(x)$ also exists, and is given by the formula

$$f'(x) = u'(v(x))v'(x).$$

The chain rule

Proof.

Define

$$\epsilon(t) = \begin{cases} \frac{u(y+t)-u(y)}{t} - u'(y) & t \neq 0 \\ 0 & t = 0 \end{cases}.$$

Note that $\epsilon(t) \to 0$ as $t \to 0$, so $\epsilon(t)$ is continuous at $t = 0$. Then $u(y+t) - u(y) = t[u'(y) + \epsilon(t)]$. Now write the difference quotient as (define $t = v(x+h) - v(x)$)

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{u(v(x+h)) - u(v(x))}{h} \\ &= \left[u'(v(x)) + \epsilon\left(v(x+h) - v(x)\right)\right] \frac{v(x+h) - v(x)}{h}. \end{aligned}$$

Letting $h \to 0$, $\epsilon\left(v(x+h) - v(x)\right) \to 0$, and the difference quotient tends to $v'(x)$, completing the proof. □

Chain rule examples

1. If $f(x) = \sin(x^2)$ then $f'(x) = \cos(x^2) \cdot 2x$.
2. If $f(x) = [v(x)]^n$ then $f'(x) = n[v(x)]^{n-1}v'(x)$.
3. If $f(x) = \sin(\sin(\sin(x)))$ then $f'(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x$.

Differentiable functions are not necessarily continuously differentiable

Consider

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Check $f'(0) = 0$ using $\left|\frac{f(h)-f(0)}{h}\right| \leq |h|$ for $h \neq 0$. Applying the chain rule for $x \neq 0$,

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

- As $x \to 0$, $2x \sin \frac{1}{x} \to 0$ but $\cos \frac{1}{x}$ does not have a limit at 0 since it takes values ± 1 at points arbitrarily close to 0.
- Since the limit of $f'(x)$ does not exist at 0, f' is not continuous there.

The derivative of an inverse function

Theorem

Let $f : [a, b] \to [c, d]$ be continuous and strictly increasing, with inverse function g. If the derivative $f'(x)$ exists and is non-zero at a point $x \in (a, b)$, then the derivative $g'(f(x))$ exists and

$$g'(f(x)) = \frac{1}{f'(x)}.$$

Proof.

Let $y = f(x)$. Given $k \neq 0$, let $h \neq 0$ solve $f(x + h) = f(x) + k$, that is, $h = g(y + k) - g(y)$. For $k \neq 0$,

$$\frac{g(y + k) - g(y)}{k} = \frac{h}{f(x + h) - f(x)}.$$

As $k \to 0$, $h \to 0$ by continuity of g at y, so that the RHS tends to $\frac{1}{f'(x)}$. □

Derivative of arctan

Use the formula $\tan'(x) = \sec^2(x)$ to obtain

$$\arctan'(x) = \cos^2(\arctan x).$$

Now write $\tan y = x$ to find $\sin^2 y = (1 - \cos^2 y) = x^2 \cos^2 y$. Hence

$$\arctan'(x) = \cos^2 y = \frac{1}{1+x^2}.$$

Derivative of arcsin

Use the formula $\sin' x = \cos x$ to obtain, for arcsin taking values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\arcsin'(x) = \frac{1}{\cos(\arcsin x)}.$$

Given $\sin y = x$, $\cos y = \sqrt{1-x^2}$, so

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Derivative of arccos

Use $\cos' x = -\sin x$ to obtain, for arccos taking values in $[0, \pi]$,

$$\arccos'(x) = -\frac{1}{\sin(\arccos x)}.$$

Given $\cos y = x$, $\sin y = \sqrt{1-x^2}$, so

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

Related rates example

Problem

Gas is pumped into a spherical balloon at a rate of 50 cc per second. If the pressure of gas in the balloon is constant, how fast is the radius of the balloon increasing when the radius is 5 cm?

Solution

Use $50 = \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$. *Since* $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2$. *Thus*

$$50 = 4\pi r^2 \frac{dr}{dt} = 100\pi \frac{dr}{dt}$$

so $\frac{dr}{dt} = \frac{1}{2\pi}$.

Implicit differentiation

An equation $F(x, y) = 0$ *implicitly* defines a curve in the plane. At points where the curve varies differentiably as a function of x or y, the equation of the tangent line can be obtained by *implicit differentiation*.

Implicit differentiation

Problem

Let C be the curve defined by the equation $3x^3 + 4x^2y - xy^2 + 2y^3 = 4$. Find the equation of the tangent line through the point $(-1, 1)$ assuming that the y coordinate varies differentiably as a function of x there.

Solution

Differentiate both sides of the equation with respect to x, to obtain

$$9x^2 + 8xy - y^2 + 4x^2y' - 2xyy' + 6y^2y' = 0 \quad \Rightarrow \quad y' = \frac{-9x^2 - 8xy + y^2}{4x^2 - 2xy + 6y^2}.$$

At $(-1, 1)$ one obtains $y' = \frac{0}{12} = 0$. Hence the tangent line has equation

$$(y - 1) = 0.$$