Math 141: Lecture 6

Applications of integrals

Bob Hough

September 19, 2016

Properties of the integral

Theorem (Linearity with respect to integrand)

If f and g are integrable on $[a, b]$, then for every pair of constants c_1, c_2, $c_1 f + c_2 g$ is integrable on $[a, b]$. Furthermore,

$$\int_a^b [c_1 f(x) + c_2 g(x)]dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx.$$

Proof.

Last class we checked the claim for $f + g$, so we'll just check the claim for cf. For each $n = 1, 2, \ldots$ choose step functions $s_n < f < t_n$,

$$\int_a^b f(x)dx - \frac{1}{n} < \int_a^b s_n(x)dx < \int_a^b t_n(x)dx < \int_a^b f(x)dx + \frac{1}{n}.$$

$\square$

Properties of the integral

Proof.

Suppose $c > 0$. Then

$$\begin{aligned}\int_a^b ct_n(x)dx - \frac{2c}{n} < \int_a^b cs_n(x)dx \\ \leq \int_a^b cf(x)dx \leq \int_a^b ct_n(x)dx.\end{aligned}$$

It follows that $0 \leq \overline{I}(cf) - \underline{I}(cf) < \frac{2c}{n}$ for every n, so the two are both equal to $\int_a^b cf(x)dx$.

The case $c < 0$ exchanges the role of lower and upper step functions. $\square$

Properties of the integral

Theorem (Additivity with respect to the interval of integration)

If two of the following three integrals exist, the third also exists, and we have

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

Proof.

Suppose both integrals on the left exist. Choose step functions s, t on $[a, b]$ with $s \leq f \leq t$ and step functions s', t' on $[b, c]$ with $s' \leq f \leq t'$ and

$$\int_a^b f(x)dx - \frac{1}{n} \leq \int_a^b s(x)dx \leq \int_a^b t(x)dx \leq \int_a^b f(x)dx + \frac{1}{n}$$

with similar inequalities for s', t' on $[b, c]$. $\square$

Properties of the integral

Proof.

Define s'' and t'' on $[a, c]$ by setting $s''(x) = s(x)$ on $[a, b]$, $s''(x) = s'(x)$ on $(b, c]$, with the corresponding definition of t''. Then $s'' \leq f \leq t''$ and

$$\begin{aligned}\int_a^b f(x)dx + \int_b^c f(x)dx - \frac{2}{n} \leq \int_a^c s''(x)dx &\leq \int_a^c t''(x)dx \\ &\leq \int_a^b f(x)dx + \int_b^c f(x)dx + \frac{2}{n}.\end{aligned}$$

This proves that the lower and upper integrals are equal to $\int_a^b f(x)dx + \int_b^c f(x)dx$. □

Properties of the integral

Proof.

Now suppose that f is integrable on $[a, b]$ and on $[a, c]$. Let s and t be lower and upper step functions for f on $[a, b]$, which approximate the integral to precision $\frac{1}{n}$ and similarly s' and t' on $[a, c]$. Define s'', t'' on $[a, b]$ by $s'' = s'$, $t'' = t'$ on $(b, c]$ and

$$s''(x) = \max(s(x), s'(x)),\ t''(x) = \min(t(x), t'(x)), \qquad x \in [a, b].$$

□

Properties of the integral

Proof.

Then

$$\int_b^c t''(x)dx - \int_b^c s''(x)dx = \int_a^c t''(x)dx - \int_a^c s''(x)dx \\ - \left(\int_a^b t''(x)dx - \int_a^b s''(x)dx\right) \leq \frac{1}{n}.$$

By taking n arbitrarily large, it follows that the upper and lower integrals of f on $[b, c]$ agree. □

Properties of the integral

Proof.

Observe that

$$\begin{aligned}\int_a^c f(x)dx - \int_a^b f(x)dx - \frac{1}{n} &\leq \int_a^c s''(x)dx - \int_a^b s''(x)dx \\ &\leq \int_a^c f(x)dx - \int_a^b f(x)dx + \frac{1}{n}\end{aligned}$$

to complete the proof. □

Properties of the integral

Theorem (Invariance under translation)

If f is integrable on $[a, b]$, then for every real c we have

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx.$$

Proof sketch.

If s and t are lower and upper step function for f on $[a, b]$, then $s(x-c)$ and $t(x-c)$ are lower and upper step functions for $f(x-c)$ on $[a+c, b+c]$ with the same integrals. It follows that the lower and upper integrals match on the two intervals. $\square$

Properties of the integral

Theorem (Expansion or contraction of the interval of integration)

If f is integrable on $[a, b]$, then for every real $k \neq 0$ we have

$$\int_a^b f(x)dx = \frac{1}{k}\int_{ka}^{kb} f\left(\frac{x}{k}\right) dx.$$

Proof.

Let $k > 0$. If s and t are lower and upper step functions for f on $[a, b]$, then $s\left(\frac{x}{k}\right)$, $t\left(\frac{x}{k}\right)$ are lower and upper step functions for $f\left(\frac{x}{k}\right)$ on $[ka, kb]$, so the theorem follows from the theorem for step functions. If $k < 0$, argue the same way, integrating on $[kb, ka]$ instead. □

Properties of the integral

Theorem (Comparison theorem)

If both f and g are integrable on $[a, b]$ and if $g(x) \leq f(x)$ for every x in $[a, b]$, then we have

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx.$$

Proof.

Let s be a lower step function for f and s' a lower step function for g. Define $s'' = \max(s, s')$, which is still a lower step function for f. Then $\int_a^b s'(x)dx \leq \int_a^b s''(x)dx$, from which it follows that

$$\int_a^b g(x)dx = \underline{I}(g) \leq \underline{I}(f) = \int_a^b f(x)dx.$$

□

The area between two integrable functions

Theorem

Assume f and g are integrable and satisfy $f \leq g$ on $[a, b]$. Then the region S between their graphs is measurable and its area $a(S)$ is given by the integral

$$a(S) = \int_a^b [g(x) - f(x)]dx.$$

The area between two integrable functions

Proof.

First suppose that both $f, g \geq 0$. Set

$$F = \{(x,y) : a \leq x \leq b, 0 \leq y < f(x)\},$$
$$G = \{(x,y) : a \leq x \leq b, 0 \leq y \leq g(x)\}.$$

Then $S = G \setminus F$. Thus $a(S) = a(G) - a(F)$. By the theorem proved last lecture regarding the area under the integral of a positive function,

$$a(G) - a(F) = \int_a^b [g(x) - f(x)]dx.$$

□

The area between two integrable functions

Proof.

If f takes negative values, choose $M > 0$ such that $f > -M$. Apply the previous calculation to $f + M$ and $g + M$ and note that this just translates the region S by M and does not change its area. The same is true of the difference of integrals. $\square$

Area under similarity

Theorem

Let $f \geq 0$ on $[a, b]$ be integrable with ordinate set S of area A. The area of kS is $k^2 A$.

Proof.

Let $g(x) = kf\left(\frac{x}{k}\right)$ on $[ka, kb]$. The ordinate set of g is the set kS. By the properties of the integral,

$$a(kS) = \int_{ka}^{kb} g(x)dx = k\int_{ka}^{kb} f\left(\frac{x}{k}\right) dx = k^2 A.$$

$\square$

Example integral

Theorem

Let $a \geq 0$, $\int_0^a x^{\frac{1}{2}} dx = \frac{2}{3} a^{\frac{3}{2}}$.

Proof.

Since the function $x^{\frac{1}{2}}$ is increasing, it is non-negative and integrable. Write S for its ordinate set on $[0, a]$, with area $a(S) = \int_0^a x^{\frac{1}{2}} dx$. Let R be the rectangle with corners at $(0, 0)$ and $(a, a^{\frac{1}{2}})$. The set $R \setminus S$ is the ordinate set minus the graph of the function $x = y^2$ on $[0, a^{\frac{1}{2}}]$, with area $\frac{a^{\frac{3}{2}}}{3}$. Thus S has area $\frac{2}{3} a^{\frac{3}{2}}$. $\square$

Trig identities

Recall that π is defined to be the area of a circle of unit radius, also, half the circumference of such a circle.

- Angles are measured in radians. The radian measure of an angle is twice the area of the sector subtended.
- The (x, y) coordinates of the subtended angle θ are $(\cos\theta, \sin\theta)$.
- For $0 < x < \frac{\pi}{2}$,
$$0 < \cos x < \frac{\sin x}{x} < 1.$$

Trig identities

Theorem

1. $\sin \frac{\pi}{2} - x = \cos x$
2. $\cos -x = \cos x$, $\sin -x = -\sin x$
3. $\cos y - x = \cos y \cos x + \sin y \sin x$.

Proof.

By Euler's formula $e^{ix} = \cos x + i \sin x$,

$$e^{i\left(\frac{\pi}{2} - x\right)} = e^{i\frac{\pi}{2}} e^{i(-x)} = \frac{i}{\cos x + i \sin x} = \sin x + i \cos x.$$

We have $e^{-ix} = \cos(-x) + i \sin -x = \frac{1}{e^{ix}} = \cos x - i \sin x$. By Euler again,

$$\begin{aligned} \cos y - x = \Re e^{i(y-x)} &= \Re(\cos y + i \sin y)(\cos x - i \sin x) \\ &= \cos y \cos x + \sin y \sin x. \end{aligned}$$

□

Trig identities

Theorem

1. $\cos a - \cos b = -2\sin\frac{a-b}{2}\sin\frac{a+b}{2}$; $\cos x$ *decreases,* $\sin x$ *increases on* $[0, \frac{\pi}{2}]$.
2. $\sin x + y = \sin x \cos y + \cos x \sin y$.

Proof.

Use $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ to write

$$-2\sin\frac{a-b}{2}\sin\frac{a+b}{2} = \frac{1}{2}\left(e^{\frac{i(a-b)}{2}} - e^{-\frac{i(a-b)}{2}}\right)\left(e^{\frac{i(a+b)}{2}} - e^{-\frac{i(a+b)}{2}}\right)$$
$$= \frac{e^{ia} + e^{-ia}}{2} - \frac{e^{ib} + e^{-ib}}{2} = \cos a - \cos b.$$

$$\sin x + y = \Im e^{i(x+y)} = \Im\left((\cos x + i\sin x)(\cos x + i\sin y)\right)$$
$$= \cos x \sin y + \sin x \cos y.$$

Trig identities

Theorem

For each $n = 1, 2, \ldots$, and $x \in \mathbb{R}$,

$$2\sin\frac{x}{2}\left(\frac{1}{2} + \sum_{k=1}^{n}\cos kx\right) = \sin\left(n + \frac{1}{2}\right)x.$$

Proof.

Use $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. The identity rearranges to

$$\left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right)\sum_{k=-n}^{n} e^{ikx} = \left(e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}\right).$$

This may be verified by induction on n. $\square$

Trig integrals

Theorem

If $0 < a < \frac{\pi}{2}$ and n is sufficiently large, we have

$$\frac{a}{n}\sum_{k=1}^{n}\cos\frac{ka}{n} < \sin a < \frac{a}{2n} + \frac{a}{n}\sum_{k=1}^{n}\cos\frac{ka}{n}.$$

Since cos is decreasing on $[0, \frac{\pi}{2}]$ it is integrable, with equal subdivision lower integrals given by

$$\frac{a}{n}\sum_{k=1}^{n}\cos\frac{ka}{n}.$$

Since the error from this lower integral is bounded by a constant times $\frac{1}{n}$, it follows that $\int_0^a \cos\theta d\theta = \sin a$.

Trig integrals

One obtains

$$\int_0^a \sin\theta d\theta = \int_0^a \cos\left(\frac{\pi}{2} - \theta\right) d\theta = \int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \cos\theta d\theta = 1 - \cos a. \qquad (1)$$

Trig integrals

Proof of inequality with sum of cosines.

By the formula two slides previous,

$$\frac{a}{n}\sum_{k=1}^{n}\cos\frac{ka}{n} = \frac{\sin(n+\frac{1}{2})\frac{a}{n} - \sin\frac{a}{2n}}{\frac{2n}{a}\sin\frac{a}{2n}}.$$

Set $\theta = \frac{a}{2n}$. By the angle addition formula,

$$\sin(2n+1)\theta = \sin 2n\theta\cos\theta + \cos 2n\theta\sin\theta < \sin 2n\theta\frac{\sin\theta}{\theta} + \sin\theta.$$

This rearranges to the first claimed inequality

$$\frac{\sin(n+\frac{1}{2})\frac{a}{n} - \sin\frac{a}{2n}}{\frac{2n}{a}\sin\frac{a}{2n}} < \sin a.$$

$\square$

Trig integrals

Proof.

To prove the second inequality, write

$$\frac{a}{2n} + \frac{a}{n}\sum_{k=1}^{n} \cos\frac{ka}{n} = \frac{\sin(n+\frac{1}{2})\frac{a}{n}}{\frac{2n}{a}\sin\frac{a}{2n}}.$$

Thus the second inequality reduces to

$$\frac{2n}{a}\sin\frac{a}{2n}\sin a < \sin a < \sin(n+\frac{1}{2})\frac{a}{n}.$$

This holds for all n large enough so that $\frac{n+\frac{1}{2}}{n}a \leq \frac{\pi}{2}$, since $\frac{\sin\theta}{\theta} < 1$, and $\sin\theta$ is increasing on $[0, \frac{\pi}{2}]$.

□

Polar coordinates

Let $f \geq 0$ on an interval $[a, b]$, where $0 \leq b - a \leq 2\pi$.

- The *radial set* of f over $[a, b]$ is the set of points in polar coordinates $\{(r, \theta) : a \leq \theta \leq b, 0 \leq r \leq f(\theta)\}$.
- If f is a constant s on interval $[a, b]$, the area of the corresponding sector is $\frac{1}{2}(b - a)s^2$.

Polar coordinates

Theorem

Let R denote the radial set of a nonnegative function f over an interval $[a, b]$, where $0 \leq b - a \leq 2\pi$, and assume that R is measurable. If f^2 is integrable on $[a, b]$ the area of R is given by the integral

$$a(R) = \frac{1}{2}\int_a^b f^2(\theta)d\theta.$$

Proof.

Let s and t be step functions with radial sets S, T and satisfying $s \leq f \leq t$. Then $S \subset R \subset T$. Hence

$$\int_a^b s^2(\theta)d\theta \leq 2a(R) \leq \int_a^b t^2(\theta)d\theta.$$

Since s^2 and t^2 are arbitrary lower and upper step functions for f^2, the claim follows. □

Average value of a function

Given n numbers $a_1, a_2, ..., a_n$, their arithmetic mean, or average, is

$$\overline{a} = \frac{1}{n} \sum_{k=1}^{n} a_k.$$

Definition

If f is integrable on an interval $[a, b]$, we define $A(f)$, the average value of f on $[a, b]$, by the formula

$$A(f) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Average value of a function

Let $w_1, w_2, ..., w_n$ be non-negative numbers, not all zero. The weighted mean of $a_1, ..., a_n$

$$\overline{a} = \frac{\sum_{k=1}^n w_k a_k}{\sum_{k=1}^n w_k}.$$

Let w be a non-negative integrable function $\int_a^b w(x) > 0$. The weighted mean of $f(x)$ on $[a, b]$

$$A(f) = \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}.$$

Metal rod example

A straight rod of length a and positive mass is positioned on the x-axis interval $[0, a]$ with integrable mass-density function $\rho(x)$. This means that the mass of the rod between b and c is $\int_b^c \rho(x)dx$.

- The *center of mass* is
$$\overline{x} = \frac{\int_0^a x\rho(x)dx}{\int_0^a \rho(x)}.$$
- The *moment of inertia* is $\int_0^a x^2\rho(x)dx$.
- The *radius of gyration* is
$$r^2 = \frac{\int_0^a x^2\rho(x)dx}{\int_0^a \rho(x)dx}.$$

Properties of indefinite integrals

Let f be a function on $[a, b]$ such that the integral $\int_a^x f(t)dt$ exists for each $x \in [a, b]$. The function

$$F(x) = \int_a^x f(t)dt, \qquad a \leq x \leq b$$

is an indefinite integral of f.
One has

$$\int_c^d f(t)dt = F(x)|_c^d = F(d) - F(c).$$

Convexity

Definition

A function f on $[a, b]$ is *convex*, resp. *concave* if, for all $x, y \in [a, b]$ and all $0 \leq \alpha \leq 1$,

$$g(\alpha y + (1-\alpha)x) \leq \alpha g(y) + (1-\alpha)g(x)$$

resp.

$$g(\alpha y + (1-\alpha)x) \geq \alpha g(y) + (1-\alpha)g(x).$$

Properties of indefinite integrals

Theorem

Let $A(x) = \int_a^x f(t)dt$. Then A is convex on every interval on which f is increasing, and concave on every interval on which f is decreasing.

Properties of indefinite integrals

Proof.

Let $x, y \in [a, b]$ with $x < y$. Let $0 < \alpha < 1$ and set $z = \alpha y + (1-\alpha)x$. Write $A(z) = (1-\alpha)A(z) + \alpha A(z)$. It suffices to show

$$(1-\alpha)[A(z) - A(x)] \le \alpha[A(y) - A(z)],$$

or, writing $z - x = \alpha(y - x)$, $y - z = (1-\alpha)(y - x)$,

$$\frac{A(z) - A(x)}{z - x} \le \frac{A(y) - A(z)}{y - z}.$$

This follows from the mean property

$$\mathrm{LHS} = \frac{1}{z - x}\int_x^z f(t)dt \le f(z) \le \frac{1}{y - z}\int_y^z f(t)dt = \mathrm{RHS}.$$

□