Math 141: Lecture 24

Equidistribution modulo 1 and related problems

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Fejér's kernel

Definition

Let $N \geq 1$. The function

$$K_N(x) = \frac{D_N(x)^2}{2N+1} = \frac{1}{2N+1}\left(\frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x}\right)^2$$

is called *Fejér's kernel*. It has Fourier coefficients

$$\hat{K}_N(n) = \begin{cases} \frac{2N+1-|n|}{2N+1} & |n| \leq 2N \\ 0 & \text{otherwise} \end{cases}.$$

Fejér's kernel

Theorem

Fejér's kernel satisfies the following properties.

1. $K_N(x) \geq 0$
2. $\int_0^1 K_N(x)dx = 1$
3. *For each fixed* $\delta > 0$, $\lim_{N \to \infty} \int_\delta^{1-\delta} K_N(x)dx = 0$.

These properties make $K_N(x)$ a 'summability kernel'.

Convergence in L^1

Theorem

Let f be integrable on $[0,1]$. Then for any $N \geq 1$,

$$\int_0^1 |f * K_N(x)| dx \leq \int_0^1 |f(x)| dx.$$

The quantity $\|f\|_1 = \int_0^1 |f(x)| dx$ is called the L^1 norm of the function f.

Convergence in L^1

Proof.

By positivity of K_N,

$$\begin{aligned}
\int_0^1 |f * K_N(x)|dx &= \int_0^1 \left|\int_0^1 f(t)K_N(x-t)dt\right| dx \\
&\le \int_0^1 \int_0^1 |f(t)|K_N(x-t)dtdx \\
&= \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \int_0^1 \int_0^1 |f(t)|e^{2\pi in(x-t)}dtdx \\
&= \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \int_0^1 e^{2\pi inx} \int_0^1 |f(t)|e^{-2\pi int}dtdx
\end{aligned}$$

The inner integral over t is a constant which depends on n but not x. Treating this as fixed we may integrate in x to eliminate all but $n = 0$, which leaves $\int_0^1 |f * K_N(x)|dx \le \int_0^1 |f(t)|dt$. □

Convergence in L^1

Theorem

Let f be integrable on $[0, 1]$. Then

$$\lim_{N\to\infty} \int_0^1 |f(x) - f * K_N(x)| dx = 0.$$

Convergence in L^1

Proof.

Given $\epsilon > 0$, choose continuous f_1 such that $\int_0^1 |f(x) - f_1(x)|dx < \frac{\epsilon}{3}$. Choose N sufficiently large so that $|f_1(x) - f_1 * K_N(x)| < \frac{\epsilon}{3}$, uniformly in x. Then

$$\begin{aligned}
&\int_0^1 |f(x) - f * K_N(x)|dx \\
&= \int_0^1 |(f - f_1)(x) + (f_1 - f_1 * K_N)(x) + ((f_1 - f) * K_N)(x)|dx \\
&\leq \int_0^1 |(f - f_1)(x)|dx + \int_0^1 |(f_1 - f_1 * K_N)(x)|dx \\
&\quad + \int_0^1 |f - f_1| * K_N(x)dx \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

□

Equidistribution modulo 1

Definition

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to *equidistribute modulo 1* if for each interval $[\alpha, \beta) \subset [0, 1)$,

$$\lim_{N\to\infty} \frac{\#\{n \leq N : a_n \bmod 1 \in [\alpha, \beta)\}}{N} = \beta - \alpha.$$

The distribution of real numbers modulo 1 and related questions plays an important role in several areas of modern analysis, including analytic number theory, probability, partial differential equations, and quantum mechanics.

Weyl's criterion

Our discussion of Fourier series permits us to prove the following famous condition of Hermann Weyl regarding equidistribution modulo 1.

Theorem (Weyl's criterion for equidistribution modulo 1)

The sequence $\{a_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if and only if, for each integer $m \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m a_n} = 0.$$

Hermann Weyl

Weyl's criterion

Proof.

First suppose that $\{a_n\}_{n=1}^\infty$ is equidistributed modulo 1. After removing the integer part, assume that $0 \leq a_n < 1$ for each n.

- Given $\epsilon > 0$, approximate the function $e^{2\pi i m x}$ with a step function

$$s(x) = \sum_{k=1}^{M} c_k 1_{[\alpha_k, \beta_k)}(x)$$

  such that for each $x \in [0,1)$, $|s(x) - e^{2\pi i m x}| < \epsilon$.
- Let N be sufficiently large so that, for each $k = 1, 2, ..., M$,

$$\left| \frac{\#\{n \leq N : a_n \in [\alpha_k, \beta_k)\}}{N} - (\beta_k - \alpha_k) \right| < \frac{\epsilon}{\sum_{k=1}^M |c_k|}.$$

$\square$

Weyl's criterion

Proof.

- Calculate

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m a_n}\right| \leq \epsilon + \left|\frac{1}{N}\sum_{n=1}^{N} s(a_n)\right|$$
$$= \epsilon + \left|\sum_{k=1}^{M} c_k \frac{\#\{n \leq N : a_n \in [\alpha_k, \beta_k)\}}{N}\right|$$
$$< 2\epsilon + \left|\sum_{k=1}^{M} c_k(\beta_k - \alpha_k)\right|$$

- Notice that $\sum_{k=1}^{M} c_k(\beta_k - \alpha_k) = \int_0^1 s(x)dx$, and $\left|\int_0^1 s(x)dx\right| \leq \left|\int_0^1 e^{2\pi i m x}dx\right| + \epsilon = \epsilon$. Hence $\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m a_n}\right| < 3\epsilon$.

□

Weyl's criterion

Proof.

Now suppose that Weyl's criterion is satisfied, that is, for each fixed integer $m \neq 0$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m a_n} = 0.$$

Given an interval $[\alpha, \beta) \subset [0, 1)$, let $f_{\alpha,\beta}$ be the function

$$f_{\alpha,\beta}(x) = \begin{cases} 1 & x \bmod 1 \in [\alpha, \beta) \\ 0 & x \bmod 1 \notin [\alpha, \beta) \end{cases}.$$

Thus

$$\frac{\#\{n \leq N : a_n \bmod 1 \in [\alpha, \beta)\}}{N} = \frac{1}{N} \sum_{n=1}^{N} f_{\alpha,\beta}(a_n).$$

$\square$

Weyl's criterion

Proof.

We claim that, for each $\epsilon > 0$ there exist *minorant* and *majorant* trigonometric polynomials $m_\epsilon(x) = \sum_{k=-K}^{K} c_k e^{2\pi ikx}$ and $M_\epsilon(x) = \sum_{k=-K}^{K} C_k e^{2\pi ikx}$ which take real values, and satisfy

- For each $x \in [0,1)$,

$$m_\epsilon(x) \le f_{\alpha,\beta}(x) \le M_\epsilon(x).$$

- $$\epsilon + \int_0^1 m_\epsilon(x)dx = \epsilon + c_0 > C_0 = \int_0^1 M_\epsilon(x)dx.$$

$\square$

Weyl's criterion

Proof.

Assuming the existence of the minorant and majorant, Weyl's proof is concluded as follows. For each N,

$$\frac{1}{N}\sum_{n=1}^{N} m_\epsilon(a_n) \leq \frac{1}{N}\sum_{n=1}^{N} f_{\alpha,\beta}(a_n) \leq \frac{1}{N}\sum_{n=1}^{N} M_\epsilon(a_n).$$

Expand the left hand side as

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{k=-K}^{K} c_k e^{2\pi i k a_n} = \sum_{k=-K}^{K} c_k \left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k a_n}\right) = c_0 + o(1)$$

as $N \to \infty$. Similarly, the right hand side is $C_0 + o(1)$ as $N \to \infty$. Since $c_0 \leq \beta - \alpha \leq C_0$ and C_0 and c_0 differ by at most ϵ, letting $\epsilon \downarrow 0$ it follows that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_{\alpha,\beta}(a_n) = \beta - \alpha$ as required. □

Majorant and minorant

Theorem

Let $0 \leq \alpha < \beta < 1$. *For each* $\epsilon > 0$ *there exist real trigonometric polynomials* $m_\epsilon(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$ *and* $M_\epsilon(x) = \sum_{n=-N}^{N} C_n e^{2\pi i n x}$ *satisfying*

- $m_\epsilon(x) \leq f_{\alpha,\beta}(x) \leq M_\epsilon(x)$
- $c_0 + \epsilon > C_0$.

Majorant and minorant

Proof.

First, choose real continuous functions g and h which are 1-periodic, such that $g(x) \leq f_{\alpha,\beta}(x) \leq h(x)$ and

$$\frac{\epsilon}{5} + \int_0^1 g(x)dx > \int_0^1 h(x)dx.$$

Choose N sufficiently large so that uniformly in x, $|g(x) - g * K_N(x)| < \frac{\epsilon}{5}$ and $|h(x) - h * K_N(x)| < \frac{\epsilon}{5}$. Set

$$m_\epsilon(x) = -\frac{\epsilon}{5} + g * K_N(x), \qquad M_\epsilon(x) = \frac{\epsilon}{5} + h * K_N(x).$$

□

Majorant and minorant

Proof.

Recall

$$m_\epsilon(x) = -\frac{\epsilon}{5} + g * K_N(x), \qquad M_\epsilon(x) = \frac{\epsilon}{5} + h * K_N(x).$$

These are real trigonometric polynomials, since the Fejér kernel is real, and satisfy $m_\epsilon(x) \le g(x) \le f_{\alpha,\beta}(x) \le h(x) \le M_\epsilon(x)$ for all x. Also,

$$\int_0^1 M_\epsilon(x)dx - \int_0^1 m_\epsilon(x)dx < \frac{4\epsilon}{5} + \int_0^1 h(x) - g(x)dx < \epsilon.$$

$\square$

Equidistribution modulo 1

Theorem

Let $\alpha \in \mathbb{R}$ be a real number. The sequence $\{\alpha n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if and only if α is irrational.

Equidistribution modulo 1

Proof.

If $\alpha = \frac{p}{q}$ is rational, then choosing $m = q$ in Weyl's criterion obtains

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i q n \alpha} = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n p} = 1$$

for all N. Hence the sequence fails to be equidistributed modulo 1. Now suppose that α is irrational. For each $m \neq 0$,

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m n \alpha}\right| = \left|\frac{1}{N}\frac{e^{2\pi i m \alpha} - e^{2\pi i m (N+1)\alpha}}{1 - e^{2\pi i m \alpha}}\right| \leq \frac{2}{N}\frac{1}{|1 - e^{2\pi i m \alpha}|}.$$

Since this tends to 0 as $N \to \infty$ the sequence is asymptotically equidistributed by Weyl's criterion. $\square$

Finite differencing

Theorem

The sequence $\{n^\alpha\}_{n=1}^\infty$ *is a sequence of integers if and only if* α *is a non-negative integer.*

Finite differencing

Proof.

- The integrality in the case α is a non-negative integer is obvious.
- If $\alpha < 0$ then n^α decreases to 0, so is not a sequence of integers.
- Define the finite difference operator on sequences by $\Delta a_n = a_{n+1} - a_n$.
- Let $m \geq 0$ satisfy $m < \alpha < m+1$. Consider the sequence

$$\{b_n = \Delta^{m+1}(n^\alpha)\}_{n=1}^\infty.$$

If $\{n^\alpha\}_{n=1}^\infty$ is a sequence of integers, then so is $\{b_n\}_{n=1}^\infty$, so it suffices to show that this is not the case.

$\square$

Finite differencing

Proof.

- Given $f(x) = x^\beta$, use Taylor's formula with remainder to write, for any $n \geq 1$ and $x \geq 0$,

$$f(x+1) - f(x) = \beta x^{\beta-1} + \binom{\beta}{2} x^{\beta-2} + \dots + \binom{\beta}{n} x^{\beta-n} + \binom{\beta}{n}(\beta - n)\int_0^1 (1-t)^n (x+t)^{\beta-n-1} dt.$$

  As $x \to \infty$, the error integral is $O(x^{\beta-n-1})$.
- Iterating this formula $m+1$ times, we find that, as $n \to \infty$,

$$b_n = (\alpha)(\alpha-1)\cdots(\alpha-m)n^{\alpha-m-1} + O(n^{\alpha-m-2}).$$

- Since $m < \alpha < m+1$, for n sufficiently large $0 < b_n < 1$.

□

Finite differencing

In fact, using differencing methods and Weyl's criterion it is possible to show that the sequence $\{n^\alpha\}_{n=1}^\infty$ becomes equidistributed modulo 1 if α is a positive non-integer real number.

Generating functions

Problem

When two standard dice are rolled, the probability distribution of the sum is distributed like the Fejér kernel:

$$p(n) = \frac{6 - |7 - n|}{36}, \qquad 2 \leq n \leq 12.$$

Find two six-sided dice A and B with positive numbers other than 1–6, which, when rolled, give the same probability distribution for the sum.

Generating functions

Solution

Represent the two standard dice with the polynomial generating function $x + x^2 + x^3 + x^4 + x^5 + x^6$*, where the coefficient on* x^i *represents the number of faces numbered* i*. Hence the counts for outcomes when the two dice are rolled is represented by*

$$\begin{aligned}&(x + x^2 + x^3 + x^4 + x^5 + x^6)^2\\ &= x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.\end{aligned}$$

Factor

$$x + x^2 + x^3 + x^4 + x^5 + x^6 = x(1 + x)(1 + x + x^2)(1 - x + x^2).$$

Generating functions

Solution

Since the dice are six sided, the coefficients should be non-negative and sum to 6. This fixes the value of the generating function at $x = 1$*. Thus each die must contain a factor of* $(1 + x)$ *and* $(1 + x + x^2)$*. To make all the numbers positive, each die must contain a factor of* x*. The only possibility is (up to exchanging their order),*

$$A(x) = x(1 + x)(1 + x + x^2)(1 - x + x^2)^2 = x + x^3 + x^4 + x^5 + x^6 + x^8,$$
$$B(x) = x(1 + x)(1 + x + x^2) = x + 2x^2 + 2x^3 + x^4.$$

Thus, one die has numbers $1, 3, 4, 5, 6, 8$ *and the other has* $1, 2, 2, 3, 3, 4$*.*