Math 141: Lecture 22

Generating functions, Fourier series

Bob Hough

November 30, 2016

Example

Recall last class we proved

Theorem

For $t > 0$,

$$\int_0^\infty e^{-tx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \arctan t.$$

by differentiating under the integral. Now we consider the behavior at $t = 0$. This is a continuous analogue of evaluating a power series at the boundary of the radius of convergence.

Example

Theorem

We have

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Example

Proof.

Integrate by parts in the integral

$$\int_A^\infty e^{-tx}\frac{\sin x}{x}dx = \frac{e^{-tA}}{A}\cos A - \int_A^\infty e^{-tx}\left(\frac{t}{x}+\frac{1}{x^2}\right)\cos x dx.$$

Bound

$$\left|\int_A^\infty e^{-tx}\frac{t}{x}\cos x dx\right| \le \frac{t}{A}\int_A^\infty e^{-tx}dx = \frac{e^{-tA}}{A},$$
$$\left|\int_A^\infty e^{-tx}\frac{\cos x}{x^2}dx\right| \le \int_A^\infty \frac{dx}{x^2} = \frac{1}{A}.$$

Thus

$$\left|\int_A^\infty e^{-tx}\frac{\sin x}{x}dx\right| \le \frac{3}{A}.$$

□

Example

Proof.

Similarly, integrate by parts in the integral

$$\int_A^\infty \frac{\sin x}{x} dx = \frac{\cos A}{A} - \int_A^\infty \frac{\cos x}{x^2} dx.$$

Thus $\left|\int_A^\infty \frac{\sin x}{x} dx\right| \leq \frac{2}{A}$. On $[0, A]$, $\frac{e^{-tx}\sin x}{x} \to \frac{\sin x}{x}$ uniformly, so

$$\begin{aligned}\frac{\pi}{2} = \lim_{t\downarrow 0} F(t) &= O(1/A) + \lim_{t\downarrow 0} \int_0^A \frac{e^{-tx}\sin x}{x} dx \\ &= O(1/A) + \int_0^A \frac{\sin x}{x} dx \\ &= O(1/A) + \int_0^\infty \frac{\sin x}{x} dx.\end{aligned}$$

Letting $A \to \infty$ completes the proof. $\square$

Infinite products

Definition

Given a sequence $\{a_n\}_{n=1}^{\infty}$ the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

is defined to be the limit, if it exists and is non-zero, of the sequence of partial products $\{P_n\}_{n=1}^{\infty}$,

$$P_n = \prod_{j=1}^{n}(1+a_j).$$

Infinite products

Theorem

Given a sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \neq -1$, whose series $\sum a_n$ is absolutely convergent, the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

is convergent. In this case we say that the product converges absolutely.

Infinite products

Proof.

Since $\sum |a_n|$ converges, $|a_n| \to 0$. Thus there is N such that $|a_n| < \frac{1}{2}$ for $n \geq N$. The sum

$$\sum_{n=N}^{\infty} \log(1 + a_n) = \sum_{n=N}^{\infty} a_n + O(a_n^2)$$

converges absolutely by comparison with $\sum_{n=N}^{\infty} |a_n|$ by using the formula $\log(1 + a_n) = a_n + O(a_n^2)$ which holds as $|a_n| \to 0$. Thus

$$\prod_{n=N}^{\infty} (1 + a_n) = \exp\left(\sum_{n=N}^{\infty} \log(1 + a_n)\right)$$

converges, as desired. □

Euler's product formula for the zeta function

Recall that the Riemann zeta function is defined in $\Re(s) > 1$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Theorem

For $\Re(s) > 1$,

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}.$$

The product, which is taken over the set of primes in increasing order, converges absolutely.

Euler's product formula for the zeta function

Proof.

- Calculate
$$\frac{1}{1-p^{-s}} - 1 = \frac{p^{-s}}{1-p^{-s}}.$$
- Set $s = x + it$ with x and t real. Then for $C = \frac{1}{1-2^{-x}} > 0$,
$$\left|\frac{1}{1-p^{-s}} - 1\right| \leq \frac{p^{-x}}{1-p^{-x}} \leq Cp^{-x}.$$
- The absolute convergence follows from
$$\sum_p \left|\frac{1}{1-p^{-s}} - 1\right| \leq C \sum_p \frac{1}{p^x} < C \sum_{n=1}^{\infty} \frac{1}{n^x} < \infty$$
which holds since $x > 1$.

$\square$

Euler's proof of the infinitude of primes

Theorem

There are infinitely many prime numbers.

Proof.

Since the sum $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\zeta(x) \to \infty$ as $x \downarrow 1$. This would be false if the product

$$\zeta(x) = \prod_p \frac{1}{1 - p^{-x}}$$

were finite. $\square$

The partition function

- Denote $p(n)$ the *partition function* of n, which counts the number of ways of writing n as the sum of one or more integers in non-increasing order.
- For instance, $p(4) = 5$ since

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

- It's conventional to define $p(0) = 1$. The first few values of p are given by $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$.

The partition function

Theorem (Hardy-Ramanujan, 1918)

As $n \to \infty$,

$$p(n) = (1 + o(1)) \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

The partition function

Theorem

For $|x| < 1$,

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

The partition function

Proof.

- The sum $\sum_{n=0}^{\infty} p(n)x^n$ converges absolutely in $|x| < 1$ by using the Hardy-Ramanujan asymptotic.
- Define
$$f_N(x) = \prod_{k=1}^{N} \frac{1}{1 - x^k} = \sum_{j=0}^{\infty} p_N(j)x^j$$
by writing $\frac{1}{1-x^k} = 1 + x^k + x^{2k} + \ldots$ and using the absolute convergence to justify the use of Cauchy products.
- The product $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$ converges absolutely to a function $f(x)$, since $\sum |x|^k$ converges.

$\square$

The partition function

Proof.

- For $j \le N$, $p_N(j) = p(j)$, by using x^{ak} to track the case that k appears a times in a partition of n in the expansion
$$\frac{1}{1-x^k} = 1 + x^k + x^{2k} + \ldots.$$
Also, $p_N(j)$ is increasing as a function of N.
- It follows that $\lim_{N\to\infty} \sum_{j=0}^\infty p_N(j)x^j$ exists, and is equal to $\sum_{j=0}^\infty p(j)x^j$, since $\sum_{j=0}^N p_N(j)x^j = \sum_{j=0}^N p(j)x^j$, and the remaining tail satisfies
$$\left|\sum_{j>N} p_N(j)x^j\right| \le \sum_{j>N} p(j)|x|^j$$
which tends to 0 as $N \to \infty$.

$\square$

Rankin's trick

A cheap version of the Ramanujan-Hardy asymptotic may be proved easily. The method of proof is known as Rankin's trick.

Theorem

For each $\delta > 0$ there is a constant $C = C(\delta) > 0$ such that $p(n) \leq C(\delta)\exp(\delta n)$.

This theorem is sufficient to obtain the convergence in $\sum_n p(n)x^n$ for $|x| < 1$ by setting $|x| = e^{\alpha}$ and choosing $2\delta = \alpha$ in the theorem to obtain

$$\sum_{n=0}^{\infty} |p(n)x^n| \leq C(\delta) \sum_{n=0}^{\infty} e^{(\delta-\alpha)n} < \infty.$$

A more careful version of the proof of the theorem gives

$$p(n) \leq \exp\left(O(\sqrt{n}\log n)\right).$$

Rankin's trick

Proof.

Choose $0 < x = e^{-\delta} < 1$ in the expression $\prod_{k=1}^n \frac{1}{1-x^k} = \sum_{j=0}^\infty p_n(j)x^j$. Since $p_n(n) = p(n)$, we can drop all but one term to obtain

$$p(n)x^n \leq \prod_{k=1}^n \frac{1}{1-x^k}.$$

Thus

$$p(n) \leq e^{\delta n} \prod_{k=1}^n \frac{1}{1-x^k} \leq e^{\delta n} \prod_{k=1}^\infty \frac{1}{1-x^k}.$$

The product converges absolutely to a constant $C(\delta)$, since $\frac{1}{1-x^k} - 1 = \frac{x^k}{1-x^k} < \frac{x^k}{1-x}$ and $\sum_k x^k$ converges. □

Fourier series

- Consider the set of functions $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$.
- These functions are 1 periodic by Euler's formula. We sometimes indicate this by saying that they are defined on $\mathbb{R}/\mathbb{Z}$.
- These functions have integral

$$\int_0^1 e^{2\pi inx}dx = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

Fourier series

- The integral formula implies the following *orthogonality relation*

$$\int_0^1 e^{2\pi imx}\overline{e^{2\pi inx}}dx = \int_0^1 e^{2\pi i(m-n)x}dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}.$$

- A *trigonometric polynomial* is a finite sum $P(x) = \sum_{n\in S} c_n e^{2\pi inx}$ where S is a finite set of frequencies.
- Calculate, using orthogonality,

$$\int_0^1 |P(x)|^2 dx = \int_0^1 \sum_{n_1,n_2\in S} c_{n_1}\overline{c_{n_2}} e^{2\pi i(n_1-n_2)x}dx = \sum_{n\in S} |c_n|^2.$$

Fourier series

Definition

Let f be integrable on $[0,1]$. The *Fourier coefficients* of f are defined by

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx.$$

The Fourier series of f is the series

$$f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}.$$

No equality is asserted by the notation $\sim$. We say that the Fourier series converges to f at the point x if

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx}.$$

Fourier series

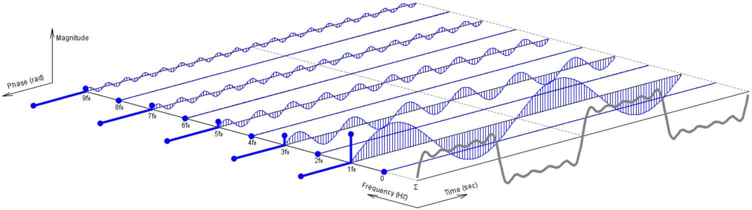

Image of Tomas Boril.

Example

Theorem

Let $s(x)$ denote the square function which is 1-periodic

$$s(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \end{cases}$$

This function has Fourier coefficients

$$\hat{s}(n) = \begin{cases} 0 & n \text{ even} \\ \frac{-2i}{\pi n} & n \text{ odd} \end{cases}.$$

Combining n and $-n$ terms,

$$s(x) \sim \frac{4}{\pi} \sum_{n>0, \text{ odd}} \frac{1}{n} \sin(2\pi n x).$$

Example

Proof.

We have

$$\hat{s}(n) = \int_0^{\frac{1}{2}} e^{-2\pi inx} dx - \int_{\frac{1}{2}}^1 e^{-2\pi inx} dx.$$

This vanishes for $n = 0$. For $n \neq 0$,

$$\hat{s}(n) = \frac{-1}{2\pi in} \left[-1 + 2e^{-\pi in} - e^{-2\pi in}\right].$$

The quantity in brackets is 0 if n even and -4 if n odd. $\square$

Dirichlet's kernel

Dirichlet's kernel is the function

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi i n x} = \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x}.$$

This satisfies

$$\int_0^1 D_N(t)dt = 1.$$

Dirichlet's kernel

The partial sums of the Fourier series of f may be expressed as

$$\begin{aligned}\int_0^1 f(t)D_N(x-t)dt &= \sum_{n=-N}^{N} e^{2\pi inx}\int_0^1 f(t)e^{-2\pi int}dt\\ &= \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx}.\end{aligned}$$

Bessel's inequality

Theorem

Let $S \subset \mathbb{Z}$ be a finite set of frequencies. For any constants $\{c_n\}_{n\in S}$,

$$\int_0^1 \left| f(x) - \sum_{n\in S} \hat{f}(n) e^{2\pi i n x} \right|^2 dx \leq \int_0^1 \left| f - \sum_{n\in S} c_n e^{2\pi i n x} \right|^2 dx.$$