Math 141: Lecture 21

Power series and applications

Bob Hough

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Continuity of power series

Theorem

Suppose $\sum_{n=0}^\infty a_n$ *converges, and define*

$$f(x) = \sum_{n=0}^\infty a_n x^n, \qquad -1 < x < 1.$$

Then $\lim_{x\to 1} f(x) = \sum_{n=0}^\infty a_n$.

Continuity of power series

Proof.

Let $s_{-1} = 0$, $s_n = \sum_{k=0}^n a_k$ and $s = \sum_{k=0}^\infty a_k$. By Abel summation,

$$\sum_{n=0}^m a_n x^n = \sum_{n=0}^m (s_n - s_{n-1}) x^n = (1-x)\sum_{n=0}^{m-1} s_n x^n + s_m x^m.$$

For $|x| < 1$, letting $m \to \infty$ obtains

$$f(x) = (1-x)\sum_{n=0}^\infty s_n x^n = s + (1-x)\sum_{n=0}^\infty (s_n - s)x^n.$$

$\square$

Continuity of power series

Proof.

Given $\epsilon > 0$, choose N such that $n > N$ implies $|s_n - s| < \epsilon$. Then

$$\sum_{n=0}^{\infty} |s_n - s||x|^n \leq \sum_{n=0}^{N} |s_n - s| + \epsilon \frac{|x|^{N+1}}{1 - |x|}.$$

Thus,

$$|f(x) - s| \leq |1 - x| \sum_{n=0}^{N} |s_n - s| + \epsilon.$$

For x sufficiently close to 1, this is bounded by 2ϵ. Letting $\epsilon \downarrow 0$ proves $\lim_{x \to 1} f(x) = s$. □

Example

- Recall that, by the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. We can now evaluate its value.
- For $|x| < 1$, $f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$. Hence, integrating,

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

- Letting $x \to 1$ obtains $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$.

Double sums

Theorem

Given a double sequence $\{a_{i,j}\}_{i,j=1}^{\infty}$ *of non-negative terms, suppose that* $\sum_{j=1}^{\infty} a_{i,j} = b_i$ *and* $\sum_{i=1}^{\infty} b_i$ *converges. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}.$$

Double sums

Proof.

Note that, for each j, $\sum_i a_{i,j}$ converges by comparison with $\sum_i b_i$. Furthermore, by the finite linearity property of convergent series,

$$\sum_{j=1}^{M}\sum_{i}^{\infty} a_{i,j} = \sum_{i=1}^{\infty}\sum_{j=1}^{M} a_{i,j} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j}.$$

Since $\sum_{j=1}^{M}\sum_{i=1}^{\infty} a_{i,j}$ is increasing as a function of M and bounded above, it converges. The equality now follows by symmetry. $\square$

Double sums

Theorem

Given a double sequence $\{a_{i,j}\}_{i,j=1}^{\infty}$, *suppose that* $\sum_{j=1}^{\infty} |a_{i,j}| = b_i$ *and* $\sum_{i=1}^{\infty} b_i$ *converges. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}.$$

Double sums

Proof.

Separate $a_{i,j}$ into its positive and negative parts. For these, the equality follows from the previous result. The claimed result now follows from finite linearity. $\square$

Change of base point

Theorem

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

the series converging in $|x| < R$. *If* $-R < a < R$, *then* f *can be expanded in a power series about the point* $x = a$, *which converges in* $|x - a| < R - |a|$, *and for* $|x - a| < R - |a|$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and $f^{(n)}(a) = \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} c_m a^{m-n}$.

Change of base point

Proof.

The formula for $f(x)$ is a general fact regarding power series. To prove the convergence, write

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n[(x-a)+a]^n \\ &= \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(x-a)^k \\ &= \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} \binom{n}{k} c_n a^{n-k}\right](x-a)^k. \end{aligned}$$

The formula for $f^{(n)}(a)$ is obtained by equating coefficients of $(x-a)^n$. $\square$

Change of base point

Proof.

The exchange in order of summation in the previous slide is justified, since

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left|c_n\binom{n}{k}a^{n-k}(x-a)^k\right| \leq \sum_{n=0}^{\infty}|c_n|(|x-a|+|a|)^n$$

converges absolutely for $|x-a|+|a|<R$. $\square$

Example

The exponential function may be rewritten as

$$\begin{aligned} E(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=n}^{\infty} \binom{k}{n} \frac{a^{k-n}}{k!} \right] (x-a)^n \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{n!} \sum_{j=0}^{\infty} \frac{a^j}{j!} \right] (x-a)^n \\ &= E(a) \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} = E(a)E(x-a). \end{aligned}$$

This gives an alternate route to prove the multiplication formula.

The logarithm

Define, for $|z| < 1$,

$$L(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}.$$

Theorem

For complex z such that $|z| < 1$,

$$E(L(1+z)) = 1 + z.$$

The logarithm

Proof.

- The equality holds for real z, since $L(1+x) = \log(1+x)$ follows by integrating the power series for $\frac{1}{1+x}$.
- We check that $E(L(1+z))$ is given by a power series in $|z| < 1$, from which the equality for complex z follows.
- Define $E_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$. For each fixed N,

$$E_N(L(1+z)) = \sum_{n=0}^{N} \frac{L(1+z)^n}{n!}$$
$$= \sum_{n=0}^{N} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^k}{k} \right)^n = \sum_{n=0}^{N} \frac{1}{n!} \left(\sum_{k=0}^{\infty} b_{k,n} z^k \right)$$

is a power series in z, obtained by taking the Cauchy product, which is justified by absolute convergence.

□

The logarithm

Proof.

- One has absolute convergence in the sum,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} b_{k,n} z^k \right) = \sum_{k=0}^{\infty} z^k \left(\sum_{n=0}^{\infty} \frac{b_{k,n}}{n!} \right)$$

by comparison with the series for

$$E(-L(1-|z|)) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \frac{|z|^k}{k} \right)^n = \frac{1}{1-|z|}$$

which is a series of only positive terms.

$\square$

Power series and differential equations

Problem

Solve the non-linear ODE $(1-x^2)y'' = -2y$.

Solution

- *Guess a solution of type* $y = \sum_{n=0}^{\infty} a_n x^n$ *with a positive radius of convergence about 0.*
- *Differentiating term-by-term*

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Power series and differential equations

Solution

- *Thus*

$$\begin{aligned}(1-x^2)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right]x^n\end{aligned}$$

- *Equating coefficients* $(n+2)(n+1)a_{n+2} - n(n-1)a_n = -2a_n$, *or*

$$a_{n+2} = \frac{n^2-n-2}{(n+2)(n+1)}a_n = \frac{n-2}{n+2}a_n.$$

Power series and differential equations

Solution

- *Since* $a_{n+2} = \frac{n-2}{n+2} a_n$, *the even coefficients are given by*

$$a_2 = -a_0, \quad a_4 = a_6 = a_8 = \ldots = 0.$$

- *The odd coefficients are given by, for* $n \geq 0$,

$$a_{2n+1} = \frac{-1}{(2n+1)(2n-1)} a_1.$$

- *Hence the full solution is given by*

$$y = a_0(1 - x^2) - a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n-1)} x^{2n+1}.$$

This convergences for $|x| < 1$, *hence is a genuine solution.*

The binomial series

Theorem

Define the generalized binomial coefficient

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

For any real α,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad |x| < 1.$$

The binomial series

Proof.

- The function $y = (1+x)^\alpha$ satisfies $y' - \frac{\alpha}{1+x} y = 0$ with initial condition $y(0) = 1$, and is the unique solution.
- Define
$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$
- The generalized binomial coefficient satisfies
$$\binom{\alpha}{n+1} = \frac{\alpha - n}{n+1} \binom{\alpha}{n},$$
and thus $f(x)$ converges in $|x| < 1$ by the ratio test.

□

The binomial series

Proof.

- Differentiating term-by-term

$$f'(x) = \sum_{n=0}^{\infty} (n+1)\binom{\alpha}{n+1} x^n.$$

- Thus

$$\begin{aligned}(1+x)f'(x) &= \sum_{n=0}^{\infty} \left\{(n+1)\binom{\alpha}{n+1} + n\binom{\alpha}{n}\right\} x^n \\ &= \alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = \alpha f(x).\end{aligned}$$

- Also, $f(0) = 1$ so $f(x) = (1+x)^{\alpha}$.

$\square$

Differentiation of integrals

Theorem

Suppose $\phi(x,t)$ is defined for $a \le x \le b$, $c \le t \le d$ and is such that the derivative $D_2\phi(x,t)$ with respect to t is a function which is uniformly continuous in both variables. Define

$$f(t) = \int_a^b \phi(x,t)dx \qquad (c \le t \le d).$$

Then for $c < t < d$, $f'(t)$ exists and

$$f'(t) = \int_a^b (D_2\phi)(x,t)dx.$$

Differentiation of integrals

Proof.

- Define the difference quotient, for $h \neq 0$,

$$\psi_h(x,t) = \frac{\phi(x,t+h) - \phi(x,t)}{h}.$$

- By the Mean Value Theorem, for each fixed x there is a u between t and $t+h$, such that

$$\psi_h(x,t) = (D_2\phi)(x,u).$$

- By the uniform continuity, for each $\epsilon > 0$ there is a $\delta > 0$, such that if $|h| < \delta$,

$$|\psi_h(x,t) - (D_2\phi)(x,t)| = |(D_2\phi)(x,u) - (D_2\phi)(x,t)| < \epsilon.$$

$\square$

Differentiation of integrals

Proof.

- Since $|\psi_h(x,t) - (D_2\phi)(x,t)| < \epsilon$,

$$\left|\int_a^b \psi_h(x,t)dx - \int_a^b (D_2\phi)(x,t)dx\right|$$
$$\leq \int_a^b |\psi_h(x,t) - (D_2\phi)(x,t)|\, dx < \epsilon(b-a).$$

- Thus $\lim_{h\to 0}\int_a^b \psi_h(x,t)dx = \int_a^b (D_2\phi)(x,t)dx$.
- Note

$$\frac{f(t+h)-f(t)}{h} = \int_a^b \psi_h(x,t)dx,$$

which completes the proof.

$\square$

Example

Theorem

For $t > 0$,

$$\int_0^\infty e^{-tx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \arctan t.$$

Example

Proof.

- Define $f(x,t) = e^{-tx}\frac{\sin x}{x}$ and set
$$F(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}dx.$$
- Thus
$$F'(t) = -\int_0^\infty e^{-tx}\sin x dx$$
by applying the previous theorem to $F_n(t) = \int_0^n e^{-tx}\frac{\sin x}{x}dx$ and letting $n \to \infty$. Note that the convergence of the derivative is uniform for t in fixed intervals $[a,b]$ with $0 < a < b$, and $F_n(t)$ converges to $F(t)$ at each point, so the theorem regarding uniform convergence of derivatives applies.

□

Example

Proof.

- Write

$$F'(t) = \frac{i}{2}\int_0^\infty e^{-tx+ix} - e^{-tx-ix}dx$$
$$= \frac{i}{2}\left[\frac{1}{t-i} - \frac{1}{t+i}\right] = \frac{-1}{1+t^2}.$$

- Thus, by the Fundamental Theorem of Calculus,

$$F(b) - F(a) = -\int_a^b \frac{dt}{1+t^2} = \arctan a - \arctan b.$$

$\square$

Example

Proof.

- Let $t \to \infty$ and observe that $\left|e^{-tx}\frac{\sin x}{x}\right| < e^{-tx}$, so

$$\lim_{t\to\infty} F(t) = 0.$$

- Thus $F(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}dx = \frac{\pi}{2} - \arctan t$.

$\square$