Math 141: Lecture 18

Sequences and infinite series

Bob Hough

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Sequences

Recall the definition of a sequence.

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is a function a whose domain is the positive integers.

Sequences are often written beginning at either index 0 or 1.

Examples

- The sequence $\{a_n = \frac{1}{n}\}_{n=1}^{\infty}$ begins
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots.$$
- The sequence of powers of 2 $\{a_n = 2^n\}_{n=0}^{\infty}$ begins
$$1, 2, 4, 8, 16, 32, 64, \ldots.$$
- The sequence $\{\sin(\pi n/2)\}_{n=1}^{\infty}$ begins
$$1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots.$$

Limits

Definition

A sequence $\{a_n\}_{n=0}^{\infty}$ has limit L if, for each $\epsilon > 0$, there is $N \geq 0$ such that $n > N$ implies

$$|a_n - L| < \epsilon.$$

A sequence $\{a_n\}_{n=0}^{\infty}$ has limit ∞ if, for each $M > 0$ there is $N \geq 0$ such that $n > N$ implies $a_n > M$.

Examples

- If $\alpha > 0$, $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.
- $\lim_{n\to\infty} x^n = \begin{cases} 0 & |x| < 1 \\ 1 & x = 1 \\ \text{Does not exist} & x = -1 \\ \infty & x > 1 \end{cases}$.
- $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$.

Monotonic sequences

Definition

A sequence $\{f(n)\}_{n=0}^{\infty}$ is *increasing* if for all n,

$$f(n+1) \geq f(n),$$

and *decreasing* if for all n,

$$f(n+1) \leq f(n).$$

The sequence is *monotonic* if it is increasing or decreasing.

Convergence of monotonic sequences

Theorem

A monotonic sequence is convergent if and only if it is bounded.

Proof.

First suppose the sequence $\{f(n)\}$ is bounded and suppose without loss of generality that it is increasing, otherwise replace $f(n)$ with $-f(n)$.

- Since $\{f(n)\}$ is bounded it has a sup, α. We'll show that this is the limit.
- Given $\epsilon > 0$ choose N such that $\alpha - f(N) < \epsilon$.
- Since f is increasing, and since α is an upper bound, for $n > N$, $\alpha - \epsilon < f(n) \leq \alpha$.

$\square$

Convergence of monotonic sequences

Proof.

Now suppose $\{f(n)\}$ has limit L. Let N be such that $n > N$ implies $|f(n) - L| < 1$. Let $M = \max\{|f(n)| : 0 \leq n \leq N\}$. It follows that for all n,

$$|f(n)| \leq \max\{|L| + 1, M\}.$$

□

Note that in the second part of the proof, monotonicity was not used. A convergent sequence is bounded.

Cauchy sequences

Definition

A sequence $\{a_n\}_{n=0}^\infty$ is *Cauchy* if, for each $\epsilon > 0$, there exists $N > 0$ such that $m, n > N$ implies $|a_n - a_m| < \epsilon$.

Examples

The sequence $\{a_n = \frac{1}{2^n}\}$ is Cauchy. To prove this, given $\epsilon > 0$, choose N such that $\frac{1}{2^N} < \frac{\epsilon}{2}$. For $m, n > N$, by the triangle inequality,

$$\left|\frac{1}{2^m} - \frac{1}{2^n}\right| \leq \frac{1}{2^m} + \frac{1}{2^n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

limsup and liminf

Definition

Let $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence. The *limit supremum* of $\{a_n\}$ is

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n > N\}.$$

The *limit infimum* of $\{a_n\}$ is

$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n > N\}.$$

Both of these limits exist.

limsup and liminf

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence. The limits supremum and infimum of a_n exist.

Proof.

- To check that the limit supremum exists, let $|a_n| \leq M$ and define sequence $\{b_N\}_{N=0}^{\infty}$ by $b_N = \sup\{a_n : n > N\}$.
- Observe that $|b_N| \leq M$ for all N, since M is an upper bound for $\{|a_n|\}$.
- The sequence $\{b_N\}$ is decreasing, since if $N < M$, the sup in b_M is taken over a subset of the set in the sup of b_N.
- Since $\{b_N\}_{N=0}^{\infty}$ is bounded and monotonic, it converges.

The argument for the limit infimum follows on replacing a_n with $-a_n$. $\square$

Completeness of the reals

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. The sequence $\{a_n\}_{n=0}^{\infty}$ has a limit if and only if it is Cauchy.

Proof.

First suppose $\{a_n\}_{n=0}^{\infty}$ converges to a limit L. Given $\epsilon > 0$, choose $N > 0$ such that $n > N$ implies $|a_n - L| < \frac{\epsilon}{2}$. Then for $m, n > N$, by the triangle inequality,

$$|a_m - a_n| = |(a_m - L) + (L - a_n)| \leq |a_m - L| + |L - a_n| < \epsilon,$$

so the sequence $\{a_n\}_{n=0}^{\infty}$ is Cauchy. □

Completeness of the reals

Proof.

Next suppose that $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence. We show that it has a limit. Observe that the Cauchy property implies that $\{a_n\}_{n=0}^{\infty}$ is bounded.

- To check this, choose $\epsilon = 1$ and let $N > 0$ be such that $m, n > N$ implies that $|a_m - a_n| < 1$.
- Thus $\{a_n\}_{n=0}^{\infty}$ is bounded by the maximum of $|a_{N+1}| + 1$ and the size of all terms preceding $|a_{N+1}|$, of which there are only finitely many.

$\square$

Completeness of the reals

Proof.

- Since $\{a_n\}_{n=0}^{\infty}$ is bounded, its limit supremum exists, call it α. We show that the limit of $\{a_n\}_{n=0}^{\infty}$ is α, also.
- Given $\epsilon > 0$, use the Cauchy property to choose N such that $n, m > N$ implies $|a_n - a_m| < \frac{\epsilon}{2}$.
- Since the sequence $b_M = \sup\{a_n : n > M\}$ decreases to limit α, we can choose $N_1 > N$ such that $M > N_1$ implies that $\alpha \leq \sup\{a_n : n > M\} < \alpha + \frac{\epsilon}{2}$.
- Choose $m > N_1$ such that $\alpha - \frac{\epsilon}{2} < a_m < \alpha + \frac{\epsilon}{2}$.
- It follows that, for $n > N$,

$$|a_n - \alpha| \leq |(a_n - a_m) + (a_m - \alpha)| \leq |a_n - a_m| + |a_m - \alpha| < \epsilon.$$

$\square$

Completeness of the reals

Definition

A metric space in which every Cauchy sequence converges to a limit is called *complete*.

Infinite series

Definition

Given a sequence $\{a_n\}_{n=1}^\infty$, its *sequence of partial sums* is the sequence $\{s_n\}_{n=1}^\infty$ defined by

$$s_n = \sum_{k=1}^n a_k.$$

The sequence $\{s_n\}_{n=1}^\infty$ is also called an *infinite series* or *series* and is denoted

$$\sum_{k=1}^\infty a_k.$$

Convergence

Definition

Given a sequence $\{a_n\}_{n=1}^{\infty}$, the infinite series $\sum_{k=1}^{\infty} a_k$ *converges* to the limit L if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ has limit L,

$$\lim_{n\to\infty} s_n = L.$$

In this case we write $\sum_{k=1}^{\infty} a_k = L$.
If $\sum_{k=1}^{\infty} a_k$ does not converge to a finite limit, it is said to *diverge*.
If $\lim_{n\to\infty} s_n = \infty$ we write $\sum_{k=1}^{\infty} a_k = \infty$ and say that the infinite series *diverges to infinity*.

Properties of convergent series

Theorem

Let $\{a_n\}_{n=1}^\infty$ *and* $\{b_n\}_{n=1}^\infty$ *be sequences of complex numbers with convergent infinite series* $\sum_{k=1}^\infty a_k$ *and* $\sum_{k=1}^\infty b_k$*. Then for any complex numbers* α *and* β*, the sequence* $\{\alpha a_n + \beta b_n\}_{n=1}^\infty$ *has a convergent infinite series, and*

$$\sum_{k=1}^\infty (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^\infty a_k + \beta \sum_{k=1}^\infty b_k.$$

Proof.

Denote $\{s_n\}_{n=1}^\infty$, $\{t_n\}_{n=1}^\infty$, $\{u_n\}_{n=1}^\infty$ the sequences of partial sums of $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and $\{\alpha a_n + \beta b_n\}_{n=1}^\infty$. Then for each n, $u_n = \alpha s_n + \beta t_n$. By the linearity property of limits,

$$\lim_{n\to\infty} u_n = \alpha \lim_{n\to\infty} s_n + \beta \lim_{n\to\infty} t_n.$$

This is the stated result. □

Properties of convergent series

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences such that $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} b_n$ diverges. Then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges. If $\sum_{n=1}^{\infty} b_n = \infty$ then $\sum_{n=1}^{\infty}(a_n + b_n) = \infty$.

Proof.

The first statement follows immediately from the previous theorem, since if $\sum_{n=1}^{\infty}(a_n + b_n)$ were convergent, this would imply the convergence of $\sum_{n=1}^{\infty} b_n$. The second statement is straightforward. $\square$

Properties of convergent series

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then $a_n \to 0$ as $n \to \infty$.

Proof.

Let $\{s_n\}_{n=1}^{\infty}$ be the sequence of partial sums. Since this sum converges, it is Cauchy. Given $\epsilon > 0$, let N be such that $n > m > N$ implies $|s_n - s_m| < \epsilon$. In particular, $|a_{m+1}| = |s_{m+1} - s_m| < \epsilon$ so $a_m \to 0$. $\square$

Examples

- $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$. To prove this, let $m < n$ and note that the sequence of partial sums satisfies

$$s_n - s_m = \sum_{k=m+1}^{n} \frac{1}{2^k} = \frac{1}{2^m} - \frac{1}{2^n}.$$

  Since this tends to 0 as a function of m the sequence is Cauchy, hence converges to a limit s. Notice that $s = 2s - s = 1$.

- $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$. To check this, note that the partial sums s_n satisfy

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = \int_1^{n+1} \frac{dx}{\lfloor x \rfloor} > \int_1^{n+1} \frac{dx}{x} = \log(n+1).$$

  Thus the sequence of partial sums tends to infinity.

- $\sum_{k=1}^{\infty} \frac{1}{2^k} + \frac{1}{k} = \infty$. This now follows from the previous theorem.

Telescoping series

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence. The sequence $\{a_n = b_n - b_{n+1}\}_{n=1}^{\infty}$ has sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ given by

$$s_n = \sum_{k=1}^{n} a_k = b_1 - b_{n+1}.$$

This series is called a telescoping series.

Telescoping series

Theorem

Let $\{b_n\}_{n=1}^{\infty}$ and $\{a_n = b_n - b_{n+1}\}_{n=1}^{\infty}$ be two sequences of complex numbers. The series $\sum a_n$ converges if and only if the sequence $\{b_n\}_{n=1}^{\infty}$ converges, in which case we have

$$\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n \to \infty} b_n.$$

Proof.

This follows from the basic properties of limits. $\square$

Example

- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. To check this, note that the sequence $\{a_n = \frac{1}{n(n+1)}\}_{n=1}^{\infty}$ satisfies

$$a_n = \frac{1}{n} - \frac{1}{n+1}.$$

The sum now follows from the telescoping property via sequence $\{b_n = \frac{1}{n}\}_{n=1}^{\infty}$.

- For x a complex number other than a negative integer, $\frac{1}{(n+x)(n+x+1)(n+x+2)} = \frac{1}{2}\left(\frac{1}{(n+x)(n+x+1)} - \frac{1}{(n+x+1)(n+x+2)}\right)$. Thus, by the telescoping property, since $\frac{1}{(n+x)(n+x+1)}$ tends to 0 as $n \to \infty$,

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)(n+x+1)(n+x+2)} = \frac{1}{2(x+1)(x+2)}.$$

Example

- The series $\sum_{n=1}^{\infty} \log \frac{n}{n+1}$ diverges to negative infinity, since $\log \frac{n}{n+1} = \log n - \log(n+1)$, and the series telescopes.

Geometric series

Theorem

If x is complex and $|x| < 1$, the geometric series $\sum_{n=0}^{\infty} x^n$ converges and

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.$$

If $|x| \geq 1$ the series diverges.

Geometric series

Proof.

The sequence of partial sums s_n satisfies

$$s_n = 1 + x + x^2 + \ldots + x^n = \begin{cases} \frac{1-x^{n+1}}{1-x} & x \neq 1 \\ n & x = 1 \end{cases}.$$

This converges, with value $\frac{1}{1-x}$, if and only if $|x| < 1$. $\square$

Examples

For $|x| < 1$,

- Replacing x with x^2 obtains

$$1 + x^2 + x^4 + ... + x^{2n} + ... = \frac{1}{1 - x^2}.$$

- Multiplying by x,

$$x + x^3 + x^5 + ... + x^{2n+1} + ... = \frac{x}{1 - x^2}.$$

- Replacing x with $-x$ obtains

$$1 - x + x^2 - x^3 + ... + (-1)^n x^n + ... = \frac{1}{1 + x}.$$

- Replacing x by x^2 obtains

$$1 - x^2 + x^4 - ... + (-1)^n x^{2n} + ... = \frac{1}{1 + x^2}.$$

Alternating series

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *alternating* if, for all $n \geq 1$, $a_n a_{n+1} \leq 0$.

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be an alternating sequence and suppose that $\{|a_n|\}_{n=1}^{\infty}$ is decreasing and tends to 0. Then $\sum_{n=1}^{\infty} a_n = L$ converges to a finite limit, and, for each $N \geq 1$,

$$\left| L - \sum_{n=1}^{N} a_n \right| \leq |a_{N+1}|.$$

Alternating series

Proof.

Without loss of generality, assume that $a_1 \geq 0$. Otherwise, multiply the sequence by -1.

- Since $|a_{2n}| \geq |a_{2n+1}| \geq |a_{2n+2}|$ and $a_{2n}, a_{2n+2} \leq 0$ while $a_{2n+1} \geq 0$, it follows that the partial sums satisfy

$$s_1 \geq s_3 \geq s_5 \geq \ldots, \qquad s_2 \leq s_4 \leq s_6 \leq \ldots,$$

  and, for all $n \geq 1$, $s_{2n-1} \geq s_{2n}$ and $s_{2n} \leq s_{2n+1}$.
- It follows that, for all $N > 2n$, $s_{2n} \leq s_N \leq s_{2n-1}$. Since the odd partial sums are decreasing and bounded below, they converge to a limit o. Since the even partial sums are increasing and bounded above they converge to a limit e.

$\square$

Alternating series

Proof.

- One has, for any n, $s_{2n} \leq e \leq o \leq s_{2n-1}$. Since $|s_{2n-1} - s_{2n}| \to 0$ as $n \to \infty$, $o = e$.
- These inequalities show that $|s_n - o| \leq |s_{n+1} - s_n| = |a_{n+1}|$ as required.

$\square$

Absolute convergence

Definition

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. If the series converges absolutely then it converges. The converse is false.

Absolute convergence

Proof.

- Let $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ denote the sequence of partial sums of $\{a_n\}_{n=1}^\infty$ and $\{|a_n|\}_{n=1}^\infty$, that is, $s_n = \sum_{k=1}^n a_k$, $t_n = \sum_{k=1}^n |a_k|$.
- Since the sequence $\{t_n\}$ converges, it is Cauchy.
- Given $\epsilon > 0$, choose N sufficiently large so that $m > n > N$ implies that $|t_m - t_n| < \epsilon$. By the triangle inequality,

$$|s_m - s_n| = \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| = |t_m - t_n| < \epsilon.$$

This proves that $\{s_n\}_{n=1}^\infty$ is Cauchy, and hence converges.

$\square$

Absolute convergence

Proof.

To prove that the converse does not hold, note that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series criteria, but does not converge absolutely, since we've already checked that $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$. $\square$

Rearrangement

Definition

A *rearrangement* of the natural numbers is a bijective map $\pi : \mathbb{N} \to \mathbb{N}$. A *rearrangement* of the sequence $\{a_n\}_{n=0}^{\infty}$ is a sequence $\{a_{\pi(n)}\}_{n=0}^{\infty}$ where π is a rearrangement of the natural numbers.

Absolute convergence

Theorem

Let $\{a_n\}_{n=0}^\infty$ *be a sequence, and suppose that* $\sum_{n=0}^\infty a_n = L$ *converges absolutely to a finite limit. Then for any rearrangement* π *of* $\mathbb{N}$*,*

$$\sum_{n=0}^\infty a_{\pi(n)} = L.$$

Absolute convergence

Proof.

- Let $\tau(n)$ denote the first number m such that $\{0, 1, 2, ..., n\} \subset \{\pi(0), \pi(1), ..., \pi(m)\}$.
- Let $\sigma(n)$ denote the maximum number among $\{\pi(0), \pi(1), ..., \pi(n)\}$.
- Given $\epsilon > 0$,
  - Since the sequence of partial sums of $|a_n|$ is Cauchy, choose N_1 such that $N_1 \leq m < n$ implies $\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}$
  - Choose N_2 such that $m \geq N_2$ implies $\left|L - \sum_{k=1}^{m} a_k\right| < \frac{\epsilon}{2}$.
  - Let $N = \max(N_1, N_2)$.
- For $n > \tau(N)$, by the triangle inequality,

$$\left|\sum_{k=0}^{n} a_{\pi(k)} - L\right| < \frac{\epsilon}{2} + \left|\sum_{k=0}^{n} a_{\pi(k)} - \sum_{k=0}^{N} a_k\right|$$

$\square$

Absolute convergence

Proof.

- Recall that $\{0, 1, 2, \ldots, N\} \subset \{\pi(0), \pi(1), \ldots, \pi(\tau(N))\}$ and $\sigma(n) = \max\{\pi(0), \ldots, \pi(n)\}$. Hence, by the triangle inequality and the Cauchy property,

$$\begin{aligned} \left| \sum_{k=0}^{n} a_{\pi(k)} - \sum_{k=0}^{N} a_k \right| &= \left| \sum_{0 \leq k \leq n, \pi(k) > N} a_{\pi(k)} \right| \\ &\leq \sum_{0 \leq k \leq n, \pi(k) > N} \left| a_{\pi(k)} \right| \\ &\leq \sum_{k=N+1}^{\sigma(n)} |a_k| < \frac{\epsilon}{2} \end{aligned}$$

It follows that $\left|\sum_{k=0}^{n} a_{\pi(k)} - L\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

□

Absolute convergence

Theorem (Riemann)

Let $\{a_n\}_{n=0}^\infty$ *be a real sequence and suppose that* $\sum_{n=0}^\infty a_n$ *is convergent, but not absolutely convergent. For any real number* α *there is a rearrangement of the natural numbers* π*, such that*

$$\sum_{n=0}^{\infty} a_{\pi(n)} = \alpha.$$

Absolute convergence

Proof.

- Denote $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \min\{a_n, 0\}$. Thus $\{a_n^+\}$ is a sequence of non-negative terms and $\{a_n^-\}$ is a sequence of non-positive terms.
- We have that $\sum_n a_n^+$ and $\sum_n a_n^-$ both diverge. To check this, note that it is impossible that one diverges and the other converges, since $\sum a_n = \sum(a_n^+ + a_n^-)$ converges. It is also impossible that both converge, since $\sum_n |a_n| = \sum_n(a_n^+ - a_n^-)$ diverges.

$\square$

Absolute convergence

Proof.

- Determine the rearrangement π as follows. If $\alpha > 0$, let $\pi(0)$ be the index of the first non-negative term in $\{a_n\}$, otherwise the index of the first negative term. For $n > 0$, if $\sum_{k=0}^{n-1} a_{\pi(k)} < \alpha$ then let $\pi(n)$ be the first unused index of a non-negative term, otherwise the first unused index of a negative term.
- Since both the sum of the non-negative and negative terms diverge, the function π alternates between taking non-negative and negative terms infinitely often, and in particular takes on all natural numbers, hence is genuinely a rearrangement.

$\square$

Absolute convergence

Proof.

- Let $s_n = \sum_{k=0}^n a_{\pi(k)}$ denote the sequence of partial sums. If $s_n - \alpha$ and $s_{n+1} - \alpha$ have the same sign, then $|s_{n+1} - \alpha| \leq |s_n - \alpha|$. If they have opposite signs, then $|s_{n+1} - \alpha| < |s_{n+1} - s_n| = |a_{\pi(n+1)}|$. In particular, for all n, $|s_n - \alpha| \leq |a_{\pi(\sigma(n))}|$ where $\sigma(n)$ is the last index before n where the sign of $s_m - \alpha$ changed.
- We have $\pi(n) \to \infty$ as $n \to \infty$ since π is a bijection, and $\sigma(n) \to \infty$ as $n \to \infty$, since there are infinitely many sign changes. Hence $\pi(\sigma(n)) \to \infty$ as $n \to \infty$.
- Since $a_n \to 0$ by convergence of $\sum a_n$, and $\pi(\sigma(n)) \to \infty$, the convergence to α follows.

$\square$