Math 141: Lecture 17

Equilibrium behavior of moving particles

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November 9, 2016

Simple harmonic motion

Recall from last class:

- Simple harmonic motion is described by the equation $y'' = -k^2 y$.
- The solutions of the equation take the form $c_1 \sin kx + c_2 \cos kx$.
- Using the trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, the general solution may be written in the form $C\sin(kx + \alpha)$ with C and α as parameters.

Force fields

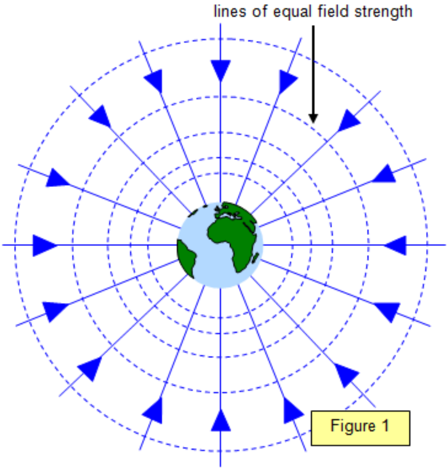
A *force field* describes the force experienced by a particle as it moves through space and time.

- We'll consider force fields which are time independent. Thus the force field is a function $F(x, x')$ which depends only upon the particle's position and possibly it's velocity.
- We assume that the particle's mass is constant. Thus Newton's second law gives $x'' = \frac{1}{m}F(x, x')$. This is a second order differential equation for position.

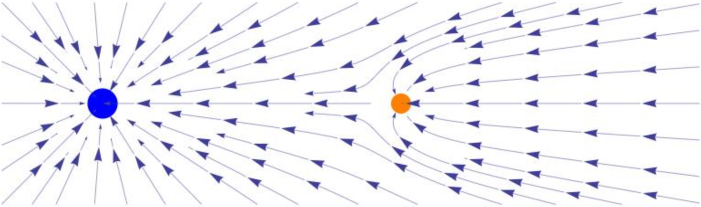
Gravity

- In Newtonian mechanics, given point masses p_1 and p_2 of masses m_1 and m_2, at distance r apart, the point masses exert a gravitational force towards each other
$$F = G\frac{m_1 m_2}{r^2}.$$
G is the gravitational constant.
- A body with spherical symmetry of its mass behaves like an equal point mass at its center.
- In problems treating free-fall near the Earth's surface, the factor r^2 is dominated by the Earth's radius, and is typically treated as a constant, so that the gravitational force is approximated as $F = gm$.

Gravity



Gravity

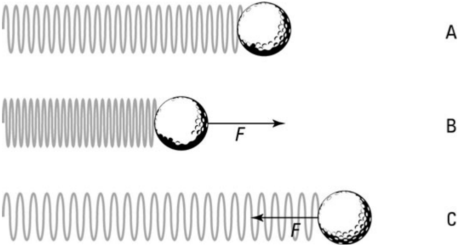


Electromagnetism

- Charged particles p_1 and p_2 at distance r, carrying charges e_1 and e_2 (signed quantities) exert an electrostatic force towards each other of
$$F = -\epsilon \frac{e_1 e_2}{r^2}$$
where ϵ is the electrostatic constant.
- The signed quantity indicates that like-charged particles repell while opposite charges attract.
- In typical experiments with charged particles, the electrostatic force overwhelms the gravitational attraction between the particles, so that gravity is ignored.

A spring

According to Hooke's Law, a mass on the end of a coiled spring experiences a force proportional and opposite the displacement of the spring from its relaxed position.



A spring

Hooke's law is an example of a general phenomenon which occurs when a system is perturbed from it's natural resting state (equilibrium). This phenomenon, simple harmonic motion, mostly explains why many physical objects have a constant vibration. Do you have a tremor?

Fields with friction

Friction of various kinds, including resistance in electric circuits, air resistance when falling, and friction when passing over a surface, is always in the direction opposite motion, and is assumed proportional to the magnitude of velocity.

Fields with friction

Equilibria

Definition

An *equilibrium point* in a force field is a point x such that $F(x, 0) = 0$.

At an equilibrium point x, the constant solution $x(t) = x$ exists for all time.

Types of equilibria

Definition

The equilibrium point x_0 in a force field F is *stable* if the trajectory $x(t)$ of a unit mass particle in F satisfies the following. For every $\epsilon > 0$ there exists $\delta > 0$ such that if at time 0, $d((x(0), x'(0)) - (x_0, 0)) < \delta$, then for all $t > 0$, $|x(t) - x_0| < \epsilon$.

Definition

The equilibrium points x_0 is *asymptotically stable* if there exists $\delta > 0$ such that if at time 0, $d((x(0), x'(0)) - (x_0, 0)) < \delta$, then $\lim_{t\to\infty} x(t) = x_0$.

Definition

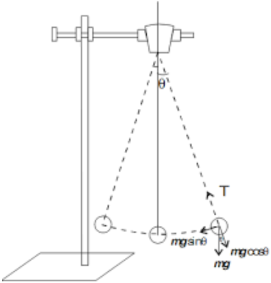
An equilibrium point which is not stable is called *unstable*.

Examples

- The field $F(x) = -k^2 x$ has an equilibrium at 0. Solutions near the equilibrium generate harmonic oscillation. The solutions are stable, but not asymptotically stable.
- The field $F(x, x') = -k_1^2 x - 2k_2^2 x'$ also has an equilibrium at 0. Solutions near the equilibrium exhibit damped harmonic oscillation. The equilibrium is asymptotically stable.

Pendulum

Consider a simple frictionless pendulum, which consists of a weightless rod with a mass (bob) at its end, constrained to rotate in a fixed vertical plane.



Pendulum

- The pendulum experiences a downward force of gravity, assumed constant, and the force of tension which keeps the weight on the end of the rod.
- At an angle θ from its downward vertical resting position, the tangential force on the pendulum is proportional to $\sin\theta$.
- The angular displacement satisfies the differential equation $\theta'' = -k \sin\theta$.

Pendulum

- The pendulum has two equilibrium points, in the upward and downward pointing directions, where the force vanishes.
- The upward pointing equilibrium is unstable, as a small displacement to either side causes the pendulum to accelerate downward.
- The downward pointing equilibrium is stable, but not asymptotically stable.
- Introducing friction into the rotation causes the stable equilibrium to become asymptotically stable.

To check these claims requires calculation.

Simple harmonic approximation

- Making the small angle approximation $\sin\theta \approx \theta$, one obtains the approximate differential equation of simple harmonic motion

$$\theta'' = -k^2\theta.$$

- This obtains the solutions $\theta(t) = C\sin(kt+\alpha)$, $\theta'(t) = Ck\cos(kt+\alpha)$.
- Given initial condition $(\theta(0), \theta'(0))$, $C > 0$ is determined by $C^2 = \theta(0)^2 + \frac{\theta'(0)^2}{k^2}$.
- The amplitude C tends to 0 as the initial conditions $\theta(0)$ and $\theta'(0)$ tend to 0.

Solution of the non-linear pendulum equation

- When the initial conditions have a small displacement from the stable equilibrium, an exact solution of the motion of the *non-linear* equation $\theta''(t) = -k^2 \sin\theta(t)$ can be given as an infinite expansion

$$\theta(t) = C\left(\sin(\tilde{k}(t+t_0)) + \epsilon_3 \sin(3\tilde{k}(t+t_0)) + \epsilon_5 \sin(5\tilde{k}(t+t_0)) + \ldots\right)$$

  where $k = \tilde{k}\left(1 + \frac{1}{4}\left(\sin^2\frac{\theta_m}{2} + \frac{3^2}{4^2}\sin^4\frac{\theta_m}{2} + \ldots\right)\right)$ and where θ_m is the maximum displacement.
- We won't treat infinite series of functions until later in the course, so we'll postpone the derivation of this result for now.
- Note that $\theta(t)$ is periodic with period $\frac{2\pi}{\tilde{k}}$, and thus the stable equilibrium is not asymptotically stable.

Behaviors near equilibria in a constant force field

Theorem

Suppose a force field $F(x)$ is twice continuously differentiable as a function of x. Let x be an equilibrium point of F.

- *If $F'(x) > 0$ then the equilibrium is unstable.*
- *If $F'(x) < 0$ then the equilibrium is stable.*

Behaviors near equilibria in a constant force field

Proof.

- The proof of the stability citerion is a little involved, but is covered in a rigorous course treating ODE's.
- To prove the instability criterion, assume without loss of generality that $x = 0$, and Taylor expand F to obtain that in a neighborhood of 0, $F(y) = F'(0)y + O(y^2)$.
- Thus there is a $\delta > 0$, such that if $0 \leq y \leq \delta$, $F(y) > \frac{F'(0)}{2}y$.
- Let $0 < x_0 < \delta$ and let $x(t)$ be the trajectory of a particle started from rest at $x(0) = x_0$ in the field F.
- Let $\tilde{x}(t)$ be the trajectory of a particle started from rest at $x(0) = \frac{x_0}{2}$ in the field $\tilde{F}(y) = \frac{F'(0)}{2}y$.

$\square$

Behaviors near equilibria in a constant force field

Proof.

- We claim that, for all t such that $0 \le x(t) \le \delta$, $x(t) > \tilde{x}(t)$.
- Suppose otherwise, and let $t_0 > 0$ be

$$t_0 = \inf\{t > 0 : x(t) < \delta \text{ and } x(t) < \tilde{x}(t)\}.$$

  For all $0 < t < t_0$, $x(t) > \tilde{x}(t)$, whence $x''(t) > \tilde{x}''(t)$ and thus, for $0 < t < t_0$, $x'(t) > x(t)$. It follows from the Mean Value Theorem that $x(t_0) > \tilde{x}(t_0)$. By continuity, $x(t) > \tilde{x}(t)$ in a neighborhood of t_0, a contradiction.
- The equation $\tilde{x}(t)$ has solution $\frac{x_0}{4}\left(e^{t\sqrt{F'(0)/2}} + e^{-t\sqrt{F'(0)/2}}\right)$, which tends to ∞ with increasing t.
- Since $\tilde{x}(t) \ge \delta$ eventually, $x(t) \ge \delta$ eventually.

$\square$

Two fixed charges

Consider two fixed positive charges on the x axis, say at $x = 1$ and $x = -1$, and a third particle with charge ϵ constrained to move along the y axis. At position y, the particle experiences a vertical force of magnitude proportional to $\frac{\epsilon y}{(1+y^2)^{\frac{3}{2}}}$. The point $y = 0$ is an equilibrium. It is stable if the particle is negatively charged, and unstable if positively charged.

Driven harmonic motion

Driven harmonic motion occurs when an external periodic force is introduced which ordinarily exhibits harmonic motion. Examples include

- A bridge that oscillates under marching soldiers.
- A tuning fork that vibrates when introduced to a sound wave.
- A child who drives a swing by pumping his legs.

Driven harmonic motion

Recall that the damped harmonic oscillation equation

$$x'' + 2ax' + b^2x = 0$$

has solutions in $0 < a < b$ given by ($d^2 = b^2 - a^2$) given by

$$x(t) = Ce^{-at}\sin(dt + \alpha),$$

where C and α are parameters. These solutions vanish in the large time limit.

Driven harmonic motion

The equation of driven harmonic motion is

$$x'' + 2ax' + b^2 x = A\cos(\omega t).$$

One guesses a particular solution of shape $B\sin(\omega t + \delta)$, since derivatives are phases of the same frequency, and adding them is a translation in time and dilation in amplitude. One can check that a solution is given by

$$\begin{aligned} x(t) &= \frac{A}{G}\sin(\omega t + \delta), \\ G &= \sqrt{(\omega^2 - b^2)^2 + 4a^2\omega^2} \\ \delta &= \cos^{-1}\frac{2a\omega}{G}. \end{aligned}$$

Driven harmonic motion

Recall

$$x(t) = \frac{A}{G}\sin(\omega t + \delta),$$
$$G = \sqrt{(\omega^2 - b^2)^2 + 4a^2\omega^2}$$

- Note that as $\omega \to b$ and $a \to 0$, $G \to 0$ so the amplitude tends to infinity. This phenomenon is called 'resonance'.
- The choice of ω which minimizes G is called the 'resonant frequency' of the system. Resonance must be considered when doing failure analysis of physical systems.