Math 141: Lecture 16

Introduction to Ordinary Differential Equations

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First order equations

Definition

A *first order differential equation* is an equation of the form

$$y' = f(x, y).$$

An *initial condition* is a condition of type $y(x_0) = y_0$.
A *solution* of this equation is a function $Y = Y(x)$ such that, for all x,

$$Y'(x) = f(x, Y(x))$$

and $Y(x_0) = y_0$.

Examples

If $f(x, y)$ is independent of y, then the differential equation is solved by integration (still not necessarily easy).

- If $y' = Q(x)$ then $y = \int Q(x)dx + C$.
- If $Y'(t) = 2\sin t$ then $Y(t) = -2\cos t + C$.

First-order linear diff eq

Definition

A *first-order linear* differential equation is an equation of form

$$y' + P(x)y = Q(x).$$

The equation is called *homogeneous* if $Q(x) = 0$.

The most famous first-order linear ode is $y' = y$, which has solution $y(x) = Ce^x$.

First-order linear diff eq

Theorem

Let P be continuous on an open interval I, and let $a \in I$. The initial value problem

$$y' + P(x)y = 0, \qquad y(a) = b$$

has the unique solution

$$y(x) = be^{-A(x)}, \qquad A(x) = \int_a^x P(t)dt.$$

Proof.

We first check that this is a solution. One has $A(a) = 0$, so $y(a) = be^0 = b$ as wanted.
Differentiation yields $y'(x) = -be^{-A(x)}A'(x) = -y(x)P(x)$ so $y'(x) + P(x)y(x) = 0$ as wanted. $\square$

First-order linear diff eq

Proof.

Suppose that $g(x)$ is another solution, solving $g'(x) + P(x)g(x) = 0$ and $g(a) = b$. Consider $h(x) = e^{A(x)}g(x)$. Then

$$h'(x) = e^{A(x)}A'(x)g(x) + e^{A(x)}g'(x) = e^{A(x)}g(x)P(x) - e^{A(x)}g(x)P(x) = 0$$

so h is a constant. Since $h(a) = b$, it follows that $g(x) = be^{-A(x)}$. □

First-order linear diff eq

Theorem

Let P and Q be continuous on an open interval I and let $a \in I$. The unique solution to the equation

$$y' + P(x)y = Q(x), \qquad f(a) = b,$$

is given by

$$f(x) = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}dt,$$

where $A(x) = \int_a^x P(t)dt$.

First-order linear diff eq

Proof.

We first check that $f(x)$ is a solution.

$$f'(x) = -P(x)be^{-A(x)} - P(x)e^{-A(x)} \int_a^x Q(t)e^{A(t)}dt + Q(x).$$

Thus $f'(x) + P(x)f(x) = Q(x)$, as wanted.
To check that the solution is unique, let $g(x)$ be a solution, and set $h(x) = e^{A(x)}g(x)$. Then

$$h'(x) = e^{A(x)}\left(P(x)g(x) + g'(x)\right) = e^{A(x)}Q(x).$$

Thus

$$h(x) = h(a) + \int_a^x e^{A(t)}Q(t)dt.$$

$\square$

Example

Problem

Find all solutions to the equation

$$xy' + (1-x)y = e^{2x}$$

on the interval $(0, \infty)$.

Example

Solution

Divide by x *to obtain the equation*

$$y' + \left(\frac{1}{x} - 1\right) y = \frac{e^{2x}}{x}.$$

Let $a = 1$ *to obtain* $A(x) = \log x - (x-1)$*. Thus a particular solution is given by*

$$e^{-A(x)} \int_1^x \frac{e^{2t}}{t} e^{A(t)} dt = \frac{e^{x-1}}{x} \int_1^x \frac{e^{2t}}{t} te^{1-t} dt = \frac{e^x}{x} \int_1^x e^t dt = \frac{e^{2x} - e^{x+1}}{x}.$$

The general solution becomes

$$f(x) = C\frac{e^x}{x} + \frac{e^{2x}}{x}$$

where C *is an arbitrary parameter.*

Radioactive decay

- A radioactive substance decays in such a way that the rate of decay is proportional to the amount present.
- Let $y = f(t)$ denote the amount of material present at time t. Thus there is a constant k such that

$$y' = -ky.$$

- The solution of this equation is $f(t) = f(0)e^{-kt}$.

Falling body in a resisting medium

- A body falling from large height experiences the downward force mg of the Earth's gravity (assumed proportional to its mass), and the upward force of air resistance $-kv$ proportional to its velocity (k is a constant).
- By Newton's second law, the velocity function $v(t)$ satisfies

$$mv' = mg - kv \quad \Leftrightarrow \quad v' + \frac{k}{m}v = g.$$

- Assuming that $v(0) = 0$, the velocity is obtained as

$$v(t) = e^{-kt/m}\int_0^t g e^{ku/m} du = \frac{mg}{k}(1 - e^{-kt/m}).$$

A cooling problem

- The rate of change in a body's temperature is proportional to the difference in its temperature and the surrounding medium.
- If $y = f(t)$ denotes the body's temperature at time t and $M(t)$ the medium's temperature then

$$y' = -k[y - M(t)] \quad \Leftrightarrow \quad y' + ky = kM(t).$$

- Thus $f(t) = be^{-kt} + e^{-kt} \int_0^t kM(u)e^{ku} du$.

Dilution

- A tank contains 100 gallons of brine of concentration 2.5 pounds of salt per gallon. Brine containing 2 pounds of salt per gallon runs into the tank at a rate of 5 gallons per minute, and the mixture (assumed uniform) pours out at the same rate.
- The salt content at time t is $y = f(t)$ (net pounds of salt) and satisfies the equation

$$y' = 10 - \frac{y}{20} \quad \Leftrightarrow \quad y' + \frac{y}{20} = 10, \qquad y(0) = 250.$$

- The solution is given by

$$y(t) = 250e^{-t/20} + e^{-t/20} \int_0^t 10e^{u/20} du = 200 + 50e^{-t/20}.$$

Circuits

- A circuit of constant inductance L and resistance R has voltage $V(t)$ and current $I(t)$ which vary with time, and satisfy the differential equation
$$LI'(t) + RI(t) = V(t).$$
- The solution to this differential equation is given by
$$I(t) = I(0)e^{-Rt/L} + e^{-Rt/L} \int_0^t \frac{V(x)}{L} e^{Rx/L} dx.$$

Constant coefficient second order equations

Definition

A *linear equation of second order* is an equation of type

$$y'' + P_1(x)y' + P_2(x)y = R(x).$$

The functions $P_1(x)$ and $P_2(x)$ are called *coefficients*. If $R(x) = 0$ the equation is *homogeneous*. If $P_1(x)$ and $P_2(x)$ are constants the equation is a *constant coefficient* equation.

Constant coefficient second order equations

- The equation $y'' = 0$ is solved by integration. All solutions take the form

$$y(x) = c_1 x + c_2$$

where c_1, c_2 are arbitrary constants.

Constant coefficient second order equations

- Constant $y'' + by = 0$, where $b < 0$. Since $b < 0$, let $b = -k^2$.
- The equation $y'' = k^2 y$ has a pair of solutions, $y = e^{kx}$, $y = e^{-kx}$.
- The general form of a solutions is $y = c_1 e^{kx} + c_2 e^{-kx}$ where c_1, c_2 are arbitrary constants (proof to come).

Constant coefficient second order equations

- Consider $y'' + by = 0$, where $b > 0$. Since $b > 0$, let $b = k^2$.
- The equation $y'' = -k^2 y$ has a pair of solutions, $y = e^{ikx}$, $y = e^{-ikx}$.
- The general form of a solutions is $y = c_1 \cos(kx) + c_2 \sin(kx)$ where c_1, c_2 are arbitrary constants. This can be rewritten in the form $y = C \sin(kx + \alpha)$ (proof to come).

Constant coefficient second order equations

- A constant coefficient equation $y'' + ay' + by = 0$ can be reduced to an equation $u'' + cu = 0$ by the substitution $u = e^{\frac{ax}{2}} y$.
- To check this, verify

$$y' = \left(u' - \frac{a}{2}u\right) e^{-\frac{ax}{2}}, \qquad y'' = \left(u'' - au' + \frac{a^2}{4}u\right) e^{-\frac{ax}{2}}.$$

- Thus $y'' + ay' + by = 0$ implies

$$\left(u'' + \left(b - \frac{a^2}{4}\right) u\right) e^{-\frac{ax}{2}} = 0 \quad \Leftrightarrow \quad u'' + \left(b - \frac{a^2}{4}\right) u = 0.$$

Uniqueness of solutions

Theorem

Let f and g be two solutions of the equation $y'' + by = 0$ on $(-\infty, \infty)$. Assume

$$f(0) = g(0), \qquad f'(0) = g'(0).$$

Then $f(x) = g(x)$ for all x.

Uniqueness of solutions

Proof.

Let $h(x) = f(x) - g(x)$ so that $h(0) = h'(0) = 0$. We show that there is an interval $I = [-\delta, \delta]$, $\delta > 0$, such that $h \equiv 0$ on I. By translation, this holds on all of $\mathbb{R}$.

To check the claim, let $M = \max_I(|h(x)|)$, and Taylor expand about 0 to obtain

$$h(x) = \int_0^x (x-t)h''(t)dt = -b\int_0^x (x-t)h(t)dt.$$

Thus

$$|h(x)| \leq M|b|\int_0^{|x|} |x-t|dt \leq \frac{M|b|x^2}{2}.$$

It follows that $M \leq M\frac{|b|\delta^2}{2}$ which forces $M = 0$ if $|b|\delta^2 < 2$. □

The Wronskian

Definition

Given a homogeneous second order equation $y'' + P_1(x)y' + P_2(x)y = 0$ and two solutions $v_1(x)$ and $v_2(x)$, their *Wronskian* is

$$W(x) = v_1(x)v_2'(x) - v_2(x)v_1'(x).$$

Constructing the particular solution

Theorem

Let v_1 and v_2 be two solutions of the homogeneous equation

$$y'' + ay' + by = 0$$

with non-vanishing Wronskian. The inhomogeneous equation

$$y'' + ay' + by = R(x)$$

has particular solution

$$y_1(x) = t_1(x)v_1(x) + t_2(x)v_2(x)$$

where

$$t_1(x) = -\int v_2(x)\frac{R(x)}{W(x)}dx, \qquad t_2(x) = \int v_1(x)\frac{R(x)}{W(x)}dx.$$

Constructing the particular solution

Proof.

Recall $t_1(x) = -\int v_2(x)\frac{R(x)}{W(x)}dx, t_2(x) = \int v_1(x)\frac{R(x)}{W(x)}dx$. Observe

$$\begin{aligned} y_1 &= t_1 v_1 + t_2 v_2 \\ y_1' &= t_1 v_1' + t_2 v_2' + (t_1' v_1 + t_2' v_2) \\ y_1'' &= t_1 v_1'' + t_2 v_2'' + (t_1' v_1' + t_2' v_2') + (t_1' v_1 + t_2' v_2)'. \end{aligned}$$

Notice $t_1'(x) = -v_2(x)\frac{R(x)}{W(x)}$, $t_2'(x) = v_1(x)\frac{R(x)}{W(x)}$. Thus $t_1' v_1 + t_2' v_2 = 0$, and $t_1' v_1' + t_2' v_2' = R(x)$. Adding the equations,

$$y_1'' + a y_1' + b y_1 = R(x).$$

$\square$

Example

Problem

Find the general solution of the equation $y'' + y = \tan x$ *on* $(-\pi/2, \pi/2)$.

Solution

Two solutions of the homogeneous equation are given by $v_1(x) = \cos x$, $v_2(x) = \sin x$. *The Wronskian is* $W(x) = v_1(x)v_2'(x) - v_2(x)v_1'(x) = \cos^2 x + \sin^2 x = 1$. *Thus*

$$t_1(x) = -\int \sin x \tan x dx = \sin x - \log|\sec x + \tan x|$$

$$t_2(x) = \int \cos x \tan x dx = \int \sin x dx = -\cos x.$$

A particular solution is thus

$$y_1 = v_1 t_1 + v_2 t_2 = -\cos x \log|\sec x + \tan x|.$$

Example

Solution

The general solution is thus

$$y = c_1 \cos x + c_2 \sin x - \cos x \log \left| \sec x + \tan x \right|.$$

Simple harmonic motion

- A particle, constrained to move in a straight line, experiences a force toward a fixed point, which is proportional to the distance from the point. This is approximated, for instance, by a releasing a stretched spring, or plucking a violin string. Absent further external forces, the particle exhibits simple harmonic motion.
- The particle's displacement from the central point is governed by

$$y'' + k^2 y = 0.$$

- This has solutions $y = A \sin kx + B \cos kx$.

Damped vibration

- A particle experiencing simple harmonic motion is damped with a external force proportional to its velocity (friction). Motion is now governed by the equation

$$y'' + 2cy' + k^2 y = 0.$$

- *Critical damping* occurs if $c^2 = k^2$. In this case, $y = e^{-cx}(A + Bx)$.
- *Overcritical damping* occurs if $c^2 > k^2$. In this case the solution has the form $y = e^{-cx}(Ae^{hx} + Be^{-hx}) = Ae^{(h-c)x} + Be^{(h+c)x}$, where $h = \sqrt{c^2 - k^2}$.

Damped vibration

- *Undercritical damping* occurs if $c^2 < k^2$. In this case,

$$y = Ce^{-cx} \sin(hx + \alpha)$$

where $h = \sqrt{k^2 - c^2}$.

Electric circuits

- An electric circuit with a capacitor satisfies the second order equation

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = V'(t).$$

- If the voltage is held constant, then

$$I''(t) + \frac{R}{L}I'(t) + \frac{1}{LC}I(t) = 0.$$

Since $\frac{R}{L} = 2c > 0$, the current tends to 0 as time tends to infinity (damping).