Math 141: Lecture 12

The Fundamental Theorem of Algebra and properties of polynomials

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Complex valued functions

Let I be an interval and let $f : I \to \mathbb{C}$ be complex valued.

- Such an f may be written as $f(x) = f_1(x) + if_2(x)$ where $f_1, f_2 : I \to \mathbb{R}$ are real valued.
- f is continuous/differentiable at a point x if and only if both f_1 and f_2 are continuous/differentiable at x. If f is differentiable at x its derivative at x is given by (the distance in the limit is the absolute value on $\mathbb{C}$)

$$f'(x) = f_1'(x) + if_2'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

- f is integrable on I if and only if both f_1 and f_2 are integrable on I. If f is integrable, its integral is given by

$$\int_a^b f(x)dx = \int_a^b f_1(x)dx + i\int_a^b f_2(x)dx.$$

Complex valued functions

- When z is a complex number, the function $f(x) = (x-z)^n$, $n \geq 0$ an integer may be expanded by the binomial theorem and has real and imaginary parts that are polynomials in x, hence are continuous and differentiable.
- When $n > 0$ is an integer, $f(x) = \frac{1}{(x-z)^n} = \frac{(x-\overline{z})^n}{(x^2-2x\Re z+|z|^2)^n}$. The denominator is a real polynomial, so continuous and differentiable, and the numerator is of the type above, so where $x \neq z$, $\frac{1}{(x-z)^n}$ is continuous and differentiable.

Complex valued functions

The formula $\frac{d}{dx}(x-z)^n = n(x-z)^{n-1}$ which is valid for all *integer* n, may be obtained by the same algebraic manipulations used to calculate the derivative in the case that z is real: e.g. for $n > 0$,

$$\begin{aligned}
&(x+h-z)^{-n} - (x-z)^{-n} \\
&= \left[\frac{1}{x+h-z} - \frac{1}{x-z}\right]\left[\sum_{j=0}^{n-1} \frac{1}{(x+h-z)^{n-1-j}(x-z)^j}\right] \\
&= \frac{-h}{(x+h-z)(x-z)}\left[\sum_{j=0}^{n-1} \frac{1}{(x+h-z)^{n-1-j}(x-z)^j}\right].
\end{aligned}$$

Thus, by continuity,

$$\lim_{h\to 0} \frac{(x+h-z)^{-n} - (x-z)^{-n}}{h} = \frac{-n}{(x-z)^{n+1}}.$$

Complex valued functions

The Fundamental Theorem of Calculus may be applied separately to the imaginary and real parts to obtain integration formulas that reverse differentiation formulas obtained.

Properties of polynomials

Theorem

Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial of degree n. For each real a, the function $p(x) = f(x+a)$ is a polynomial of degree n.

Proof.

By the Binomial Theorem,

$$p(x) = f(x+a) = \sum_{k=0}^{n} c_k \sum_{j=0}^{k} \binom{k}{j} x^j a^{k-j}$$

$$= \sum_{j=0}^{n} x^j \left[\sum_{k=j}^{n} c_k \binom{k}{j} a^{k-j} \right].$$

The bracketed quantity is a constant, which makes $p(x)$ a polynomial. □

Complex numbers

Recall the following properties of complex numbers.

- A complex number has form $z = a + bi$ where a and b are real.
- It's modulus is $r = |z| = \sqrt{a^2 + b^2}$ and its angle is $\theta = \tan^{-1} \frac{b}{a}$.
- Euler's formula is the representation $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$
- To multiply two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, multiply their moduli and add their angles,

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

- The number $e^{i\theta} = \cos\theta + i\sin\theta$ has modulus 1, since $\cos^2\theta + \sin^2\theta = 1$.
- The complex conjugate of z is $\overline{z} = a - ib = re^{-i\theta}$. Complex conjugation commutes with arithmetic:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}.$$

The Fundamental Theorem of Algebra

Theorem (The Fundamental Theorem of Algebra)

Let $P(z)$ be a complex polynomial of degree $n \geq 1$. The equation $P(z) = 0$ has a solution in $\mathbb{C}$.

Proof.

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ where $a_n \neq 0$.

- Define $\mu = \inf \{|P(z)| : z \in \mathbb{C}\}$.
- When $|z| = R$ with $R > 1$ one has

$$\begin{aligned} |P(z)| &\geq |a_n| R^n - (|a_{n-1}| R^{n-1} + ... + |a_0|) \\ &\geq |a_n| R^n \left(1 - \frac{|a_{n-1}| + ... + |a_0|}{|a_n| R}\right). \end{aligned}$$

- Thus there is some $R > 0$ such that for $|z| > R$, $|P(z)| > \mu + 1$.

The Fundamental Theorem of Algebra

Proof.

- Since $|P(z)|$ is continuous on $B_R(0) = \{z \in \mathbb{C} : |z| \leq R\}$, the infimum μ is achieved by a point z_0 with $|z_0| \leq R$, see HW8.
- Suppose $P(z_0) \neq 0$. We then examine the behavior of $P(z)$ near z_0 to reach a contradiction.
- Define $Q(z) = \frac{P(z_0+z)}{P(z_0)}$, which is a polynomial in z with constant term equal to 1.
- Write $Q(z) = 1 + b_k z^k + b_{k+1} z^{k+1} + ... + b_n z^n$ with $b_k \neq 0$ the lowest order coefficient not equal to 0, besides the constant term.

The Fundamental Theorem of Algebra

Proof.

- Recall $Q(z) = \frac{P(z_0+z)}{P(z_0)} = 1 + b_k z^k + b_{k+1} z^{k+1} + \ldots + b_n z^n$.
- Let $b_k = r_k e^{i\theta_k}$ with $r_k > 0$. Consider $z = re^{i(\frac{\pi}{k} - \frac{\theta_k}{k})}$. Thus $b_k z^k = r_k r^k e^{i\pi} = -r_k r^k$.
- Write, for $0 < r < 1$,

$$|b_{k+1}z^{k+1} + \ldots + b_n z^n| \leq r^{k+1}(|b_{k+1}| + \ldots + |b_n|).$$

  Thus, if r is sufficiently small, then $|b_{k+1}z^{k+1} + \ldots + b_n z^n| \leq \frac{1}{2} r_k r^k$.
- In particular, for such r,

$$|Q(z)| \leq 1 - r_k r^k + \frac{1}{2} r_k r^k \leq 1 - \frac{1}{2} r_k r^k < 1$$

  which implies that $|P(z_0 + z)| < |P(z_0)|$, a contradiction.

It follows that $P(z_0) = 0$. □

Properties of polynomials

Theorem

Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial of degree at most n with coefficients in a field $\mathbf{F}$. If $f(x) = 0$ for $n+1$ distinct values of x, then every coefficient c_k is 0 and $f(x) = 0$ for all x.

Proof.

The proof is by induction on n.

- Base case: $n = 0$. In this case, $f(x) = c$ is a constant. Evaluating at the root, $c = 0$, so $f(x) = 0$ for all x.
- Inductive step: Suppose the claim holds for some $n = k \geq 0$. Let $n = k + 1$ be the degree of f, and let r be one of the roots. By the division algorithm for polynomials,

$$f(x) = (x - r)f_1(x) + b.$$

Properties of polynomials

Proof.

Recall $f(x) = (x - r)f_1(x) + b$.

- Evaluating at $x = r$, $b = 0$.
- Since the degree of f_1 is at most k and f_1 vanishes at all roots of f aside from r, $f_1(x) = 0$ for all x, so $f(x) = 0$ for all x, and all of its coefficients vanish.

$\square$

Properties of polynomials

Theorem

Let $\mathbf{F}$ *be a field. Given distinct field elements* $x_1, x_2, ..., x_n$ *and (possibly equal) values* $a_1, ..., a_n$*, there is a unique polynomial* $P(x)$ *with coefficients in* $\mathbf{F}$*, of degree at most* $n-1$ *satisfying for* $1 \leq i \leq n$, $P(x_i) = a_i$*.*

Proof.

Two such polynomials have a difference which vanishes at n points, hence vanishes entirely by the previous theorem. Thus it suffices to prove the existence.

Properties of polynomials

Proof.

To prove existence:

- Let $Q_1(x), ..., Q_n(x)$ be defined by

$$Q_i(x) = \frac{\prod_{1\leq j\leq n, j\neq i}(x - x_j)}{\prod_{1\leq j\leq n, j\neq i}(x_i - x_j)}.$$

- Note that $Q_i(x)$ is a polynomial of degree $n-1$, and satisfies $Q_i(x_i) = 1$ and for $j \neq i$, $Q_i(x_j) = 0$.
- Define $P(x) = \sum_{i=1}^n a_i Q_i(x)$, which satisfies the condition.

□

Partial fractions over $\mathbb{C}$

Theorem

Let $P(z)$ and $Q(z) \neq 0$ be complex polynomials without common roots. Let

$$Q(z) = (z - z_1)^{e_1} \ldots (z - z_n)^{e_n}$$

where $e_1, \ldots, e_n$ are positive integer exponents. The rational function $R(z) = \frac{P(z)}{Q(z)}$ has a unique expression as

$$R(z) = p(z) + \frac{a_{1,1}}{z - z_1} + \ldots + \frac{a_{1,e_1}}{(z - z_1)^{e_1}} + \ldots + \frac{a_{n,1}}{z - z_n} + \ldots + \frac{a_{n,e_n}}{(z - z_n)^{e_n}},$$

where $p(z)$ is a polynomial and the $a_{i,j}$ are complex number coefficients.

Remark: This is a general set-up, since the Fundamental Theorem of Algebra guarantees that a complex polynomial with leading coefficient 1 has an expression as a product of linear terms $(z - z_i)$ where the z_i are roots. The leading coefficient of Q can be pushed into P.

Partial fractions over $\mathbb{C}$

Proof.

- Consider the polynomial equation

$$P(z) = Q(z)\Bigg[p(z) +$$

$$+ \frac{a_{1,1}}{z - z_1} + \ldots + \frac{a_{1,e_1}}{(z - z_1)^{e_1}} + \ldots + \frac{a_{n,1}}{z - z_n} + \ldots + \frac{a_{n,e_n}}{(z - z_n)^{e_n}}\Bigg].$$

- Setting, successively, $z = z_1, z_2, \ldots, z_n$ determines the values of $a_{1,e_1}, \ldots, a_{n,e_n}$, since exactly one term on the right does not vanish in each case.
- Let

$$P_1^*(z) = P(z) - Q(z)\left[\frac{a_{1,e_1}}{(z - z_1)^{e_1}} + \frac{a_{2,e_2}}{(z - z_2)^{e_2}} + \ldots + \frac{a_{n,e_n}}{(z - z_n)^{e_n}}\right].$$

Partial fractions over $\mathbb{C}$

Proof.

- Note that $P_1^*(z)$ vanishes at $z_1, ..., z_n$. Cancel a factor of $(z - z_1)...(z - z_n)$ from $P_1^*(z)$ obtaining $P_1(z)$, and from $Q(z)$ obtaining $Q_1(z)$. This produces the equation,

$$P_1(z) =$$
$$Q_1(z)\left[p(z) + \frac{a_{1,1}}{z - z_1} + ... + \frac{a_{1,e_1-1}}{(z - z_1)^{e_1-1}} + ... + \frac{a_{n,e_n-1}}{(z - z_n)^{e_n-1}}\right].$$

- Iterate this process (formally, apply induction) m stages until all negative power terms on the right have been eliminated. Since $Q_m(z) = 1$, this obtains the equation $P_m(z) = p(z)$ which determines $p(z)$.
- Since each of the coefficients is determined in this process, the representation is unique.

$\square$

Partial fractions over $\mathbb{R}$

Theorem

Let $P(x)$ and $Q(x)$ be polynomials. The rational function $R(x) = \frac{P(x)}{Q(x)}$ may be expressed as a linear combination of functions of the following types:

1. *Polynomials*
2. *Negative integer powers of a linear factor:* $\frac{1}{(x-r)^n}$
3. *Negative integer powers of an irreducible quadratic factor:* $\frac{1}{((x-a)^2+b)^n}$, $b > 0$.
4. *Negative integer powers of an irreducible quadratic factor, with derivative in the numerator:* $\frac{2x-2a}{((x-a)^2+b)^n}$, $b > 0$.

Partial fractions over $\mathbb{R}$

Proof sketch.

- Initially perform the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ over $\mathbb{C}$, as described above.
- If z is a complex root of $Q(x)$ then $\overline{z}$ is also a root, since $Q(\overline{z}) = \overline{Q(z)} = 0$. Dividing off a factor of $(x-z)(x-\overline{z})$, which is real, then repeating the argument if necessary, it follows that $(x-z)$ and $(x-\overline{z})$ appear as factors of $Q(x)$ an equal number of times.
- By taking complex conjugates throughout the partial fraction decomposition procedure, which leaves $P(x)$ and $Q(x)$ unchanged, it follows that for each j, $\frac{1}{(x-z)^j}$ and $\frac{1}{(x-\overline{z})^j}$ appear with coefficients that are complex conjugate of each other.

Partial fractions over $\mathbb{R}$

Proof sketch.

- The polynomial $(x - z)(x - \overline{z}) = x^2 - 2\Re(z)x + |z|^2$ is quadratic irreducible. Form a common denominator in

$$\sum_{j=1}^{m} \frac{c_j}{(x-z)^j} + \frac{\overline{c_j}}{(x-\overline{z})^j} = \frac{p(x)}{(x^2 - 2x\Re(z) + |z|^2)^m}.$$

  The polynomial $p(x)$ is real, since the left hand side is invariant under complex conjugation.

- One can obtain a decomposition of $\frac{p(x)}{(x^2-2x\Re(z)+|z|^2)^m}$ into terms of type $\frac{ax+b}{(x^2-2x\Re(z)+|z|^2)^j}$ by performing repeated long divisions.

$\square$

Convex hull

Definition

Let $x_1, x_2, \ldots, x_n$ be n points of $\mathbb{R}^2$. The *convex hull* of $x_1, \ldots, x_n$ is the set

$$C = \{t_1 x_1 + t_2 x_2 + \ldots + t_n x_n : 0 \leq t_1, \ldots, t_n \text{ and } t_1 + \ldots + t_n = 1\}.$$

The convex hull of a set of points is convex, in the sense that, if $a, b \in C$, then the line segment $(1-t)a + tb$, $0 \leq t \leq 1$ connecting a and b is contained in C.

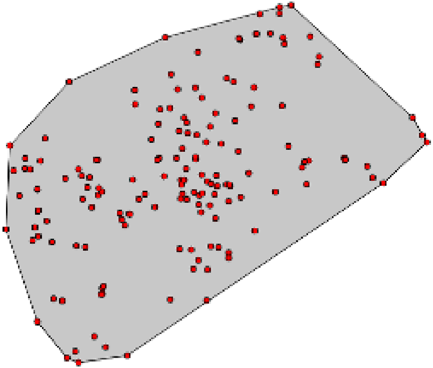
Convex hull

Definition

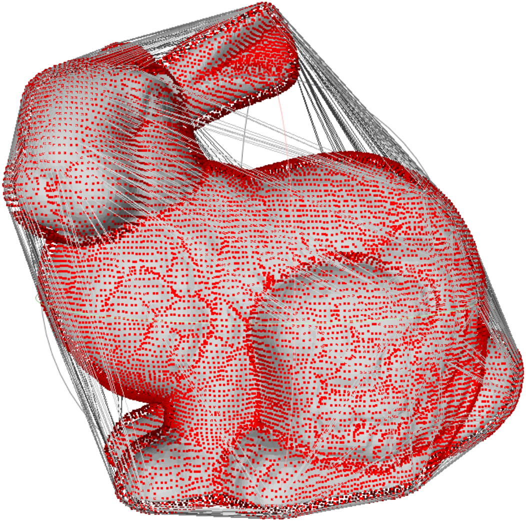
Given n points $x_1, \ldots, x_n$ of $\mathbb{R}^2$, a supporting line of $x_1, \ldots, x_n$ is a line ℓ such that all n points lie on the same side of ℓ.

An equivalent definition of the convex hull is the intersection of all half-planes containing the points. The line defining such a half-plane is a supporting line.

Convex hull example



Convex hull example in 3d



The Gauss-Lucas Theorem

Theorem (Gauss-Lucas Theorem)

Let $P(z)$ be a complex polynomial. The roots of $P'(z)$ lie within the closed convex hull of the set of roots of $P(z)$.

Proof.

By the Fundamental Theorem of Algebra, $P(z) = a\prod_{j=1}^{n}(z - z_j)$ where $a \neq 0$ and $z_1, ..., z_n$ are the roots of P. Then

$$\frac{P'}{P}(z) = \sum_{j=1}^{n} \frac{1}{z - z_j}.$$

The Gauss-Lucas Theorem

Proof.

- Let ℓ be a supporting line for $z_1, \ldots, z_n$ in $\mathbb{C}$.
- Let $w = e^{i\theta}z + b$ be a rotation and translation of $\mathbb{C}$ so that $z \in \ell$ if and only if w is real. Let $Q(w) = P(z)$ so that the roots $w_1, \ldots, w_n$ of Q all lie on one side of the x-axis, say have positive imaginary part.
- Let w have negative imaginary part. Then for each j, $w - w_j$ has negative imaginary part, and hence $\frac{1}{w-w_j} = \frac{\overline{w-w_j}}{|w-w_j|^2}$ has positive imaginary part.
- It follows that for w of negative imaginary part, $\frac{Q'}{Q}(w) = \sum_{j=1}^n \frac{1}{w-w_j}$ has positive imaginary part, hence is non-zero.
- By the chain rule, $Q'(e^{i\theta}z + b)e^{i\theta} = P'(z)$, and hence all zeros of P' lie on the same side of ℓ as the zeros of P. Since this is true for all ℓ, all zeros of P' are within the convex hull of the zeros of P.

$\square$