Math 141: Lecture 11

The Fundamental Theorem of Calculus and integration methods

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First Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus)

Let f be a function that is integrable on $[a,b]$. Let c be such that $a \leq c \leq b$ and define, for $a \leq x \leq b$,

$$A(x) = \int_c^x f(t)dt.$$

The derivative $A'(x)$ exists at each point $x \in (a,b)$ where f is continuous and $A'(x) = f(x)$.

First Fundamental Theorem of Calculus

Proof.

- Let f be continuous at x. Given $\epsilon > 0$ choose $\delta > 0$ such that if $|h| < \delta$, $|f(x+h) - f(x)| < \epsilon$.
- Write, for $h \neq 0$,

$$A(x+h) - A(x) = \int_c^{x+h} f(t)dt - \int_c^x f(t)dt = \int_x^{x+h} f(t)dt.$$

  Thus

$$\frac{A(x+h) - A(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt = f(x) + \frac{1}{h}\int_x^{x+h} (f(t) - f(x))dt.$$

- For $|h| < \delta$ the last integral is bounded in size by ϵ, from which the limit follows.

$\square$

Zero-derivative Theorem

Theorem

If $f'(x) = 0$ for each x in an open interval I, then f is constant on I.

Proof.

Let $x \neq y$ in I. Since f is differentiable, by the Mean Value Theorem there is z between x and y such that $0 = f'(z) = \frac{f(y)-f(x)}{y-x}$. $\square$

Second Fundamental Theorem of Calculus

Theorem (Second Fundamental Theorem of Calculus)

Let f be continuous on (a, b) and suppose $P'(x) = f(x)$ for all $x \in (a, b)$. Then for all $c \in (a, b)$,

$$P(x) = P(c) + \int_c^x f(t)dt.$$

Proof.

Observe that $Q(x) = P(x) - \int_c^x f(t)dt$ has $Q'(x) = 0$, and thus $Q(x) = C$ is a constant. Set $x = c$ to find $C = Q(c) = P(c)$. $\square$

Substitution Theorem for Integrals

Theorem

Assume g has a continuous derivative g' on an open interval I. Let f be continuous on the range of g. Then for each $c, x \in I$,

$$\int_c^x f[g(t)]g'(t)dt = \int_{g(c)}^{g(x)} f(u)du.$$

Proof.

Define

$$F(x) = \int_c^x f[g(t)]g'(t)dt - \int_{g(c)}^{g(x)} f(u)du.$$

Then by the first FTC and the chain rule

$$F'(x) = f[g(x)]g'(x) - f[g(x)]g'(x) = 0,$$

so F is constant on I. Evaluate at $x = c$ to find $F(x) = 0$. $\square$

Examples

Problem

Integrate $\int x^3 \cos x^4 dx$.

Solution

Substitute $u = x^4$, *using* $du = 4x^3 dx$. *Thus the integral is given by* $\frac{1}{4} \sin x^4 + C$.

Examples

Problem

Integrate $\int \cos^2 x \sin x dx$.

Solution

Substitute $u = \cos x$, *so that* $du = -\sin x dx$. *Thus*

$$\int \cos^2 x \sin x dx = -\int u^2 du = -\frac{\cos^3 x}{3} + C.$$

Examples

Problem

Integrate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$.

Solution

Let $u = \sqrt{x}$ *so that* $du = \frac{1}{2\sqrt{x}}dx$. *Hence*

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin u du = -2\cos\sqrt{x} + C.$$

Examples

Problem

Integrate $\int \frac{xdx}{\sqrt{1+x^2}}$.

Solution

Substitute $u = 1 + x^2$, $du = 2xdx$. *Hence*

$$\int \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = u^{\frac{1}{2}} + C = \sqrt{1+x^2} + C.$$

Examples

Problem

Evaluate $\int_2^3 \frac{(x+1)dx}{\sqrt{x^2+2x+3}}$.

Solution

Substitute $u = x^2 + 2x + 3,\ du = 2x + 2$ *to find*

$$\int_2^3 \frac{(x+1)dx}{\sqrt{x^2+2x+3}} = \frac{1}{2}\int_{11}^{18} u^{-\frac{1}{2}} du = \sqrt{u}\Big|_{11}^{18} = \sqrt{18} - \sqrt{11}.$$

Integration by parts

Theorem

Let f and g be continuously differentiable on $[a, b]$. Then

$$\int_a^b f(t)g'(t)dt = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t)dt.$$

Proof.

Write $h(x) = f(x)g(x)$. Then $h'(x) = f(x)g'(x) + f'(x)g(x)$ and hence

$$h(b) - h(a) = \int_a^b h'(x)dx = \int_a^b f(x)g'(x) + f'(x)g(x)dx.$$

Rearranging these integrals gives the claim. $\square$

Examples

Problem

Integrate $\int x \cos x dx$.

Solution

Let $u = x$ *and* $dv = \cos x dx$ *so* $du = dx$ *and* $v = \sin x$. *Integrating by parts*

$$\int x \cos x dx = x \sin x - \int \sin x dx + C = x \sin x + \cos x + C.$$

Examples

Problem

Integrate $\int x^2 \cos x dx$.

Solution

Let $u = x^2$ *and* $dv = \cos x dx$ *so that* $du = 2x dx$ *and* $v = \sin x$. *Integrating by parts obtains*

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx + C.$$

Now set $u = x$, $dv = \sin x dx$ *so* $du = dx$ *and* $v = -\cos x$ *and integrate by parts again to obtain*

$$\int x \sin x dx = -x \cos x + \sin x + C$$
$$\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

Definition of the natural logarithm

Definition

The natural logarithm $\log : (0, \infty) \to \mathbb{R}$ is defined by

$$\log x = \int_1^x \frac{dt}{t}.$$

Basic properties of the logarithm

Theorem

The natural logarithm satisfies the following properties.

1. $\log(1) = 0$.
2. $\log'(x) = \frac{1}{x}$.
3. $\log(ab) = \log(a) + \log(b)$ *for every* $a > 0, b > 0$.

Proof.

The first item is immediate, and the second follows from the FTC. For the third, write

$$\log(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}.$$

In the second integral, make the substitution $u = \frac{t}{a}$, $du = \frac{dt}{a}$, so that the second integral becomes $\int_1^b \frac{dt}{t}$. Thus $\log(ab) = \log(a) + \log(b)$. $\square$

Properties of the functional equation

Theorem

The natural logarithm $\log : (0, \infty) \to \mathbb{R}$ *is a bijection.*

Proof.

- Since $\log'(x) = \frac{1}{x} > 0$, log is strictly increasing, hence injective.
- To check that it is surjective, note that $\log \frac{1}{x} = -\log x$, so it suffices to prove that log takes every positive real value.
- In fact, log is differentiable, hence continuous, so by the Intermediate Value Theorem, it suffices to check that $\log x$ is unbounded.
- Observe, for $n \geq 0$,

$$\int_{2^n}^{2^{n+1}} \frac{dt}{t} \geq \frac{1}{2^{n+1}} \int_{2^n}^{2^{n+1}} dt = \frac{1}{2}.$$

It follows that $\log(2^n) \geq \frac{n}{2}$ for all n.

$\square$

Logarithms to other bases

Definition

Let $b > 0$, $b \neq 1$ be a real number. The logarithm base b is the function $\log_b : (0, \infty) \to \mathbb{R}$ defined by

$$\log_b x = \frac{\log x}{\log b}.$$

The power function

- Let $b > 0$, $b \neq 1$. Note that $\log_b b = 1$.
- Hence, for every $n \in \mathbb{Z}$, $\log_b b^n = n$ by the addition property of the logarithm.
- Similarly, for any $n \geq 0$, $\log_b b^{\frac{1}{n}} = \frac{1}{n}$, and hence for all rational $\frac{p}{q}$, $\log_b b^{\frac{p}{q}} = \frac{p}{q}$.

Definition

Let $b > 0$, $b \neq 1$ and let $r \in \mathbb{R}$. Define b^r to be the unique real number such that $\log_b b^r = r$. Define $1^r = 1$.

The exponential function

Definition

Euler's constant e satisfies $\log e = 1$.

Theorem

The exponential function $e^x : \mathbb{R} \to (0, \infty)$ *is the inverse function of* $\log$. *It satisfies the following properties:*

1. $e^0 = 1$, $e^1 = e$.
2. $\frac{d}{dx} e^x = e^x$.
3. *For all real* a, b, $e^{a+b} = e^a e^b$.

Proof.

The first and third properties follow from those of log. By the chain rule

$$1 = \frac{d}{dx} \log(e^x) = \frac{1}{e^x} \frac{d}{dx} e^x.$$

□

Integrals involving logs

When $a < b < 0$, substitute $u = -t$, $du = -dt$ to find

$$\int_a^b \frac{dt}{t} = \int_{-a}^{-b} \frac{du}{u} = \log|b| - \log|a|.$$

Thus $\int \frac{dt}{t} = \log|x| + C$, with the proviso that the range of integration does not include 0.

Integrals involving logs

Problem

Integrate $\int \tan x dx$.

Solution

Since $\tan x = \frac{\sin x}{\cos x}$, *set* $u = \cos x$, $du = -\sin x dx$ *to find*

$$\int \tan x dx = -\int \frac{du}{u} = -\log|u| + C = -\log|\cos x| + C.$$

Integrals involving logs

Problem

Integrate $\int \log x dx$.

Solution

Let $u = \log x$, $dv = dx$, $du = \frac{dx}{x}$, $v = x$ *and integrate by parts to find*

$$\int \log x dx = x \log x - \int 1 dx = x \log x - x + C.$$

Integrals involving logs

Problem

Integrate $\int \sin(\log x)dx$.

Solution

Let $u = \sin(\log x)$, $dv = dx$, $du = \frac{\cos(\log x)}{x}$, $v = x$ *and integrate by parts*

$$\int \sin(\log x)dx = x\sin(\log x) - \int \cos(\log x)dx.$$

Let $u = \cos(\log x)$, $dv = dx$, $du = -\frac{\sin(\log x)}{x}$, $v = x$ *and integrate by parts*

$$\int \cos(\log x)dx = x\cos(\log x) + \int \sin(\log x)dx.$$

Adding the equations, $2\int \sin(\log x)dx = x\sin(\log x) - x\cos(\log x) + C$.

Logarithmic differentiation

Theorem

Let f be differentiable in a neighborhood of x and satisfy $f(x) \neq 0$. Then $\frac{d}{dx} \log |f(x)| = \frac{f'(x)}{f(x)}$.

Proof.

This follows from the Fundamental Theorem of Calculus and the chain rule. $\square$

Examples

Logarithmic differentiation aids in calculating the derivative of products.

Problem

Calculate $f'(x)$ *if* $f(x) = x^2 \cos x (1 + x^4)^{-7}$.

Solution

Let $g(x) = \log|f(x)| = 2\log|x| + \log|\cos x| - 7\log(1 + x^4)$. *Then*

$$g'(x) = \frac{2}{x} - \tan x - \frac{28x^3}{1 + x^4},$$

so

$$f'(x) = f(x)g'(x) = x^2 \cos x (1 + x^4)^{-7} \left(\frac{2}{x} - \tan x - \frac{28x^3}{1 + x^4}\right).$$

Partial fractions

Theorem

Let $P(x)$ and $Q(x)$ be polynomials. The rational function $R(x) = \frac{P(x)}{Q(x)}$ may be expressed as a linear combination of functions of the following types:

1. *Polynomials*
2. *Negative integer powers of a linear factor:* $\frac{1}{(x-r)^n}$
3. *Negative integer powers of an irreducible quadratic factor:* $\frac{1}{((x-a)^2+b)^n}$, $b > 0$.
4. *Negative integer powers of an irreducible quadratic factor, with derivative in the numerator:* $\frac{2x-2a}{((x-a)^2+b)^n}$, $b > 0$.

We will discuss the proof of this theorem after proving the Fundamental Theorem of Algebra.

Partial fractions

By making linear translations of the type $u = x - a$, $u = ax$ the problem of integrating rational functions reduces to integrals of the following type

1. x^{-1} with $\int \frac{dx}{x} = \log|x| + C$.
2. x^n, $n \neq -1$ an integer, with $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.
3. $\frac{2x}{x^2+1}$ with $\int \frac{2x}{x^2+1} dx = \log|x^2+1| + C$
4. For $n > 1$ an integer, $\frac{2x}{(x^2+1)^n}$, with $\int \frac{2x}{(x^2+1)^n} = \frac{(x^2+1)^{1-n}}{1-n} + C$.
5. $\frac{1}{x^2+1}$ with $\int \frac{dx}{x^2+1} = \tan^{-1} x + C$.
6. For $n > 1$ an integer, $\frac{1}{(x^2+1)^n}$. This is most easily treated using complex numbers, see the next slide.

Partial fractions

- Factor $\frac{1}{x^2+1} = \frac{1}{(x-i)(x+i)} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$.
- Using this formula repeatedly, one may express, for each $n = 1, 2, \ldots$,

$$\frac{1}{(x^2+1)^n} = \sum_{j=1}^{n} c_{j,n}\left(\frac{1}{(x-i)^j} + \frac{(-1)^j}{(x+i)^j}\right).$$

  For instance, $\frac{1}{(x^2+1)^2} = \frac{-1}{4}\left(\frac{1}{(x-i)^2} + \frac{1}{(x+i)^2} - \frac{2}{(x-i)(x+i)}\right) = \frac{-1}{4}\left(\frac{1}{(x-i)^2} + \frac{1}{(x+i)^2} - \frac{1}{i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)\right)$.
- For $j > 1$, use the formula $\int \frac{dx}{(x-\alpha)^j} = \frac{1}{1-j}\frac{1}{(x-\alpha)^{j-1}} + C$, which remains valid for α complex.

Examples

Problem

Integrate $\int \frac{x^2+2x+3}{(x-1)(x+1)^2}dx$.

Solution

- *First, solve for*

$$\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2} \qquad \Leftrightarrow$$
$$x^2+2x+3 = A_1(x+1)^2 + A_2(x-1)(x+1) + A_3(x-1).$$

- *Set* $x = 1$ *to find* $A_1 = \frac{3}{2}$. *Set* $x = -1$ *to find* $A_3 = -1$. *By considering the* x^2 *term,* $A_2 = \frac{-1}{2}$.
- *Thus*

$$\int \frac{x^2+2x+3}{(x-1)(x+1)^2}dx = \frac{3}{2}\log|x-1| - \frac{1}{2}\log|x+1| + \frac{1}{x+1} + C.$$

Examples

Problem

Integrate $\int \frac{3x^2+2x-2}{x^3-1}dx$.

Solution

- *Factor* $x^3 - 1 = (x-1)(x^2+x+1) = (x-1)((x+1/2)^2 + \frac{3}{4})$.
- *Solve for* A, B, C *such that*

$$\frac{3x^2+2x-2}{x^3-1} = \frac{A}{x-1} + \frac{B(2x+1)+C}{x^2+x+1} \qquad \Leftrightarrow$$
$$3x^2+2x-2 = A(x^2+x+1) + (B(2x+1)+C)(x-1).$$

- *Set* $x = 1$ *to find* $A = 1$.
- *Thus* $2x^2 + x - 3 = (B(2x+1)+C)(x-1)$, *so* $B(2x+1)+C = 2x+3$ *and* $B = 1, C = 2$.

Examples

Solution

Hence

$$\int \frac{3x^2+2x-2}{x^3-1}dx = \log|x-1| + \log(x^2+x+1) + 2\int \frac{dx}{(x+1/2)^2+3/4}.$$

To evaluate the last integral, set $u = \sqrt{\frac{4}{3}}(x+1/2)$, $dx = \frac{\sqrt{3}}{2}du$ *to obtain*

$$\begin{aligned} 2\int \frac{dx}{(x+1/2)^2+3/4} &= \sqrt{3}\int \frac{du}{3/4(u^2+1)} = \frac{4}{\sqrt{3}}\tan^{-1} u + C \\ &= \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + C. \end{aligned}$$

The inequality between the arithmetic and geometric means

Theorem

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then

$$G_n = (x_1 \cdot \ldots \cdot x_n)^{\frac{1}{n}} \leq \frac{x_1 + \ldots + x_n}{n} = A_n.$$

In words, the geometric mean is less than or equal to the arithmetic mean.

Proof.

Let $f(x) = \log x$ on $(0, \infty)$. Then $f''(x) = -\frac{1}{x^2} < 0$, so f is concave. By Jensen's inequality,

$$\log(A_n) \geq \frac{\log(x_1 \cdot \ldots \cdot x_n)}{n},$$

and, exponentiating, $A_n \geq G_n$. □

Complex valued functions

Let I be an interval and let $f : I \to \mathbb{C}$ be complex valued.

- Such an f may be written as $f(x) = f_1(x) + if_2(x)$ where $f_1, f_2 : I \to \mathbb{R}$ are real valued.
- f is continuous/differentiable at a point x if and only if both f_1 and f_2 are continuous/differentiable at x. If f is differentiable at x its derivative at x is given by (the distance in the limit is the absolute value on $\mathbb{C}$)

$$f'(x) = f_1'(x) + if_2'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

- f is integrable on I if and only if both f_1 and f_2 are integrable on I. If f is integrable, its integral is given by

$$\int_a^b f(x)dx = \int_a^b f_1(x)dx + i\int_a^b f_2(x)dx.$$

Complex valued functions

- When z is a complex number, the function $f(x) = (x - z)^n$, $n \geq 0$ an integer may be expanded by the binomial theorem and has real and imaginary parts that are polynomials in x, hence are continuous and differentiable.
- When $n > 0$ is an integer, $f(x) = \frac{1}{(x-z)^n} = \frac{(x-\overline{z})^n}{(x^2-2x\Re z+|z|^2)^n}$. The denominator is a real polynomial, so continuous and differentiable, and the numerator is of the type above, so where $x \neq z$, $\frac{1}{(x-z)^n}$ is continuous and differentiable.

Complex valued functions

The formula $\frac{d}{dx}(x-z)^n = n(x-z)^{n-1}$ which is valid for all *integer* n, may be obtained by the same algebraic manipulations used to calculate the derivative in the case that z is real: e.g. for $n > 0$,

$$\begin{aligned}
&(x+h-z)^{-n} - (x-z)^{-n} \\
&= \left[\frac{1}{x+h-z} - \frac{1}{x-z}\right]\left[\sum_{j=0}^{n-1} \frac{1}{(x+h-z)^{n-1-j}(x-z)^j}\right] \\
&= \frac{-h}{(x+h-z)(x-z)}\left[\sum_{j=0}^{n-1} \frac{1}{(x+h-z)^{n-1-j}(x-z)^j}\right].
\end{aligned}$$

Thus, by continuity,

$$\lim_{h\to 0} \frac{(x+h-z)^{-n} - (x-z)^{-n}}{h} = \frac{-n}{(x-z)^{n+1}}.$$

The Fundamental Theorem of Calculus can now be applied to the imaginary and real parts to obtain integration formulas that reverse the differentiation.