MATH 141, FALL 2016 PRACTICE MIDTERM 1

SEPTEMBER 28

Solve 4 of 6 problems. You may quote any result stated during lecture, so long as you represent the result accurately.

Problem 1. Prove by induction

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Solution. The proof is by induction.

Base case: $n = 0$. The sum is empty, hence equal to 0, which is also the value of the RHS.

Inductive step: Suppose for some $n \geq 0$ that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 &= (n+1)^2 \left(\frac{n^2}{4} + n + 1\right) \\
&= \left(\frac{n+1}{2}\right)^2 (n^2 + 4n + 4) \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2.
\end{aligned}$$

This completes the inductive step.

Problem 2. Given a function f on $\mathbb{N}$, we say $\lim_{n\to\infty} f(n) = A$ if, for every $\epsilon > 0$ there exists $N > 0$ such that $n > N$ implies $|f(n) - A| < \epsilon$. Evaluate

$$\lim_{n\to\infty} n^{-3/2} \sum_{k=1}^{n} \sqrt{k}.$$

Solution. The limit has value $\frac{2}{3}$.

To justify this, observe that the function $f(x) = \sqrt{x}$ is increasing on $[0, 1]$. Let s_n and t_n denote the lower and upper step functions for $f(x)$ obtained by partitioning $[0, 1]$ into n equal subintervals and choosing the initial value of each subinterval for s_n and the final value of each subinterval for t_n. By the proof from lecture that increasing functions are integrable,

$$\int_0^1 s_n(x)dx \le \int_0^1 \sqrt{x}dx = \frac{2}{3} \le \int_0^1 t_n(x)dx \le \int_0^1 s_n(x)dx + \frac{\sqrt{1} - \sqrt{0}}{n}.$$

Now write

$$n^{-3/2} \sum_{k=1}^{n} \sqrt{k} = \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{k}{n}} = \int_0^1 t_n(x)dx.$$

Thus, for each $n = 1, 2, ...,$ $\frac{2}{3} \le \int_0^1 t_n(x)dx \le \frac{2}{3} + \frac{1}{n}$, and thus, given $\epsilon > 0$ in the condition for the limit, the requirement is met by taking $N = \frac{1}{\epsilon}$.

Problem 3. Prove that no order can be defined in the complex field that turns it into an ordered field.

Solution. We first make an observation: Let $x \neq 0$ be an element of an ordered field. Then $x \cdot x$ is positive. Indeed, either x or $-x$ is positive, whence $x \cdot x = (-x) \cdot (-x)$ is positive.

Suppose for contradiction that $\mathbb{C}$ is ordered. Then both $1 = (-1) \cdot (-1)$ and $-1 = i \cdot i$ are positive, contradiction.

Problem 4. A complex number z is said to be *algebraic* if there are integers $a_0, a_1, ..., a_n$, not all 0, such that

$$a_0 + a_1 z + \cdots + a_n z^n = 0.$$

Prove that the set of algebraic numbers is countable. Is every real number algebraic?

Solution. Let $\mathcal{P}$ denote the set of non-zero polynomials with integral coefficients. Given a non-zero polynomial P, denote $r(P)$ the set of roots of P. The set Alg of algebraic numbers is

$$\mathrm{Alg} = \bigcup_{P \in \mathcal{P}} r(P).$$

Since the set of roots of a non-zero polynomial P is finite, hence countable, and the countable union of countable sets is countable, it suffices to prove that $\mathcal{P}$ is countable.

By mapping $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, $a_n \neq 0$ to the tuple $(a_n, a_{n-1}, ..., a_0) \in \mathbb{Z}^{n+1}$ we obtain an injective map $\mathcal{P} \mapsto \bigcup_{n \geq 1} \mathbb{Z}^n$. It therefore suffices to check that $S = \bigcup_{n \geq 1} \mathbb{Z}^n$ is countable. Since the collection of sets in the union is countable, it suffices to check that $\mathbb{Z}^n$ is countable for every $n \geq 1$. This we prove by induction.

In lecture we constructed injections $f_1 : \mathbb{Z} \to \mathbb{N}$ and $f_2 : \mathbb{Z}^2 \to \mathbb{N}$. Suppose $n \geq 2$ and that we have an injection $f_n : \mathbb{Z}^n \to \mathbb{N}$. Define a map $f_{n+1} : \mathbb{Z}^{n+1} \to \mathbb{N}$ by writing $\mathbb{Z}^{n+1} = \mathbb{Z}^n \times \mathbb{Z}$, and $x \in \mathbb{Z}^{n+1}$ as $x = (x_1, x_2)$ with $x_1 \in \mathbb{Z}^n, x_2 \in \mathbb{Z}$. The map f_{n+1} is

$$f_{n+1}(x_1, x_2) = f_2(f_n(x_1), x_2).$$

Note that $(x_1, x_2) \mapsto (f_n(x_1), x_2)$ is an injection $\mathbb{Z}^{n+1} \to \mathbb{Z}^2$ since $(f_n(x_1), x_2) = (f_n(x_1'), x_2')$ implies $x_2 = x_2'$ and $x_1 = x_1'$ (since f_n is an injection). Being the composition of injective functions, f_{n+1} is injective, completing the proof.

There exist non-algebraic real numbers, since the set of real numbers is uncountable, whereas the set of real algebraic numbers is a subset of the set of all algebraic numbers, hence countable.

Problem 5. Let $A_a^b(f)$ denote the average of integrable function f on an interval $[a,b]$. Suppose that f is integrable on every sub-interval of $[a,b]$. If $a < c < b$, prove that there is a number t satisfying $0 < t < 1$ such that $A_a^b(f) = tA_a^c(f) + (1-t)A_c^b(f)$. Thus A_a^b is a weighted average of A_a^c and A_c^b.

Solution. Set $t = \frac{c-a}{b-a}$ so that $1 - t = \frac{b-c}{b-a}$. We check that $A_a^b(f) = tA_a^c(f) + (1-t)A_c^b(f)$ as follows.

$$\begin{aligned}
A_a^b(f) &= \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{b-a}\left[\int_a^c f(x)dx + \int_c^b f(x)dx\right] \\
&= \frac{c-a}{b-a}\frac{1}{c-a}\int_a^c f(x)dx + \frac{b-c}{b-a}\frac{1}{b-c}\int_c^b f(x)dx \\
&= tA_a^c(f) + (1-t)A_c^b(f).
\end{aligned}$$

Problem 6. Give the proof of the following theorem from lecture. Let f be a continuous function, and suppose $f(c) > 0$. Then there is a neighborhood $N(c, \delta)$ such that $f(x) > 0$ for all $x \in N(c, \delta)$.

Solution. f is continuous at c means that, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $x \in N(c, \delta)$ implies $|f(c) - f(x)| < \epsilon$. Choose $\epsilon = \frac{|f(c)|}{2}$ to obtain such a δ. Then for $x \in N(c, \delta)$, if $f(c) > 0$,

$$f(c) - f(x) < \frac{f(c)}{2} \qquad \Rightarrow \qquad f(x) > \frac{f(c)}{2} > 0$$

while if $f(c) < 0$,

$$f(x) - f(c) < -\frac{f(c)}{2} \qquad \Rightarrow \qquad f(x) < \frac{f(c)}{2} < 0.$$

In either case, for all $x \in N(c, \delta)$, $f(x)$ has the same sign as $f(c)$.