MATH 141, FALL 2016 PRACTICE FINAL

DECEMBER 15

Solve 6 of 8 problems. You may quote any result stated during lecture, so long as you represent the result accurately.

Problem 1.

a. (2 points) State carefully the definition of the supremum of a set which is bounded above.

b. (3 points) Prove that a sequence which is increasing and bounded above converges to it's supremum.

Solution. a. Let S be a set which is bounded above. The supremum s of S is the unique number such that s is an upper bound for S, and any other upper bound b of S satisfies $b \geq s$.

b. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence which is bounded above. Let s denote it's supremum. Since s is a supremum, given $\epsilon > 0$ there is an N such that $a_N > s - \epsilon$, otherwise $s - \epsilon$ would be a smaller lower bound. Since the sequence is increasing, for $n > N$, $a_n \geq a_N > s - \epsilon$. Also, since s is an upper bound $a_n \leq s$. Thus, for $n > N$, $|s - a_n| < \epsilon$, so $\{a_n\}_{n=1}^{\infty}$ converges to s.

Problem 2.

a. (2 points) State the Intermediate Value Theorem.

b. (3 points) Let $f : [0,1] \to [0,1]$ be continuous. Prove that there is $c \in [0,1]$ such that $f(c) = c$.

Solution. a. Let f be continuous on the closed interval $[a,b]$. For each value s between $f(a)$ and $f(b)$ there is an $x \in [a,b]$ such that $f(x) = s$.

b. Let $g(x) = f(x) - x$, which is continuous on $[0,1]$. Since $g(0) \geq 0$ and $g(1) \leq 0$ the equation $g(x) = 0$ has a solution $x \in [0,1]$. This x satisfies $f(x) = x$.

Problem 3.

a. (2 points) Give the definition of an uniformly continuous function on a closed interval $[a, b]$.
b. (3 points) Give the proof from lecture that a continuous function on a closed interval $[a, b]$ is bounded.

Solution. a. A function f is uniformly continuous on $[a, b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
b. See Lecture 7, slides 28–29.

Problem 4. Evaluate the following limits.

a. (3 points)

$$\lim_{n\to\infty}\left(\frac{2n!}{n!\cdot n^n}\right)^{\frac{1}{n}}.$$

b. (2 points)

$$\lim_{x\to 0}\frac{\log(1+x)-x}{1-\cos x}.$$

Solution. a. Let $s=\lim_{n\to\infty}\left(\frac{2n!}{n!\cdot n^n}\right)^{\frac{1}{n}}$. Write

$$\frac{2n!}{n!n^n}=\prod_{k=1}^{n}\left(1+\frac{k}{n}\right).$$

Thus

$$\begin{aligned}\log s &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{k}{n}\right)\\ &= \int_1^2 \log x dx\\ &= x\log x - x\Big|_1^2 = 2\log 2-1.\end{aligned}$$

Hence $s=\frac{4}{e}$.

b. Write $\log(1+x)-x=-\frac{x^2}{2}+O(x^3)$ and $1-\cos x=\frac{x^2}{2}+O(x^3)$. Hence the limit is

$$\lim_{x\to 0}\frac{-\frac{x^2}{2}+O(x^3)}{\frac{x^2}{2}+O(x^3)}=-1.$$

Problem 5. Determine whether each series converges. If the series converges, determine whether it converges absolutely.

a. (3 points)

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right).$$

b. (2 points)

$$\sum_{n=1}^{\infty}\frac{1-n\sin(1/n)}{n}.$$

Solution. a. Write

$$\begin{aligned} a_n=\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right) &= \prod_{k=1}^{n}\left(1-\frac{1}{2k}\right) \\ &= \exp\left(\sum_{k=1}^{n}\frac{-1}{2k}+O(1/k^2)\right) \\ &= \exp\left(-\frac{\log n}{2}+O(1)\right)=\frac{\exp(O(1))}{\sqrt{n}}. \end{aligned}$$

Since a_n decreases monotonically to 0, the series converges by the alternating series test. It does not converge absolutely by comparison with the series $\sum \frac{1}{\sqrt{n}}$ (or by Gauss's test).

b. Expand $\sin(1/n)=1/n-\frac{1}{6n^3}+O(1/n^5)$. Hence the series is

$$\sum_{n=1}^{\infty}\left(\frac{1+O(1/n^2)}{6n^3}\right)$$

which converges absolutely.

Problem 6.

a. (2 points) Determine the degree 5 Taylor polynomial of the function $f(x) = \sin x \cos x^2$.

b. (3 points) Determine the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n! z^n}{n^n}$.

Solution. a. Write $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)$ and $\cos x^2 = 1 - \frac{x^4}{2} + O(x^8)$. Since both series converge absolutely their product is given by the Cauchy product, which is

$$\sin x \cos x^2 = x - \frac{x^3}{6} + \left[-\frac{1}{2} + \frac{1}{120} \right] x^5 + O(x^7).$$

The degree 5 Taylor polynomial is thus

$$x - \frac{x^3}{6} - \frac{59}{120} x^5.$$

b. Let $a_n = \frac{n!}{n^n}$. Then $\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ as $n \to \infty$. It follows by the ratio test that the radius of convergence is e.

Problem 7.

a. (2 points) Give the definition of a sequence of functions f_n which converges uniformly to a function f on the interval $[a, b]$.

b. (3 points) Prove that the sequence of partial sums of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on every closed interval $[a, b] \subset \mathbb{R}$.

Solution. a. $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $[a, b]$ if, for each $\epsilon > 0$ there exists N such that $n > N$ implies, for all $x \in [a, b]$, $|f_n(x) - f(x)| < \epsilon$.

b. Let $M = \max(|a|, |b|)$. Let $a_n = \frac{M^n}{n!}$. Since $\sum_{n=0}^{\infty} a_n = e^M$ converges, the uniform convergence follows by the Weierstrass M-test.

Problem 8.

a. (3 points) Let $f(x) = (x - 1/2)^2$ on $[0, 1]$. Calculate the Fourier coefficients $\hat{f}(n)$ in the Fourier series of f.

b. (2 points) Prove that the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$ converges uniformly to $f(x)$ on $[0, 1]$.

Solution. a. We have $\hat{f}(0) = \int_0^1 (x - \frac{1}{2})^2 dx = 2\int_0^{\frac{1}{2}} x^2 dx = \frac{1}{12}$. For $n \neq 0$, integrating by parts twice,

$$\begin{aligned}
\hat{f}(n) &= \int_0^1 \left(x - \frac{1}{2}\right)^2 e^{-2\pi inx} dx \\
&= \left.\frac{(x - \frac{1}{2})^2 e^{-2\pi inx}}{-2\pi in}\right|_0^1 + \frac{1}{2\pi in}\int_0^1 (2x - 1)e^{-2\pi inx} dx \\
&= \frac{1}{\pi in}\int_0^1 xe^{-2\pi inx} dx \\
&= \left.\frac{xe^{-2\pi inx}}{2\pi^2 n^2}\right|_0^1 - \frac{1}{2\pi^2 n^2}\int_0^1 e^{-2\pi inx} dx = \frac{1}{2\pi^2 n^2}.
\end{aligned}$$

b. Since $\hat{f}(n) = \hat{f}(-n)$ and these terms are paired together, the Fourier series is given by

$$\tilde{f}(x) = \frac{1}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^2}.$$

Let $M_n = \frac{1}{n^2}$. Since $\sum_n M_n < \infty$, the uniform convergence follows from the Weierstrass M-test. By the uniform convergence, $\tilde{f}(x)$ is continuous. Also, the uniform convergence guarantees that

$$\hat{\tilde{f}}(n) = \int_0^1 \tilde{f}(x)e^{-2\pi inx} dx = \hat{f}(n).$$

Since f and $\tilde{f}$ are continuous functions with equal Fourier coefficients, they are equal, e.g. since $\int_0^1 |f(x) - \tilde{f}(x)|^2 dx = 0$.