MATH 141, FALL 2016 MIDTERM 2

NOVEMBER 2

Solve 4 of 6 problems. You may quote results stated during lecture, so long as you represent the result accurately.

Problem 1.

a. (2 points) State carefully the definition of a function differentiable at a point a.

b. (3 points) Let $\alpha > 1$. If $|f(x)| \leq |x|^\alpha$, prove that f is differentiable at 0. Let $0 < \beta < 1$. Prove that if $|f(x)| \geq |x|^\beta$ and $f(0) = 0$, then f is not differentiable at 0.

Problem 2. Suppose that $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. Prove *Leibniz's formula*:

$$(f \cdot g)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) \cdot g^{(n-k)}(a).$$

Problem 3.

a. (2 points) State the weighted (non-integral) Jensen's inequality.

b. (3 points) Using Jensen's inequality, or otherwise, prove the following *Power Mean Inequality.* Let $x_1, x_2, ..., x_n \in \mathbb{R}_{>0}$. If $0 < a < b$ then

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^a\right)^{\frac{1}{a}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^b\right)^{\frac{1}{b}}.$$

(Hint: set $y_i = x_i^a$ to reduce to the case $a = 1$.)

Problem 4. Use integration by parts to derive the recursion formula for $n \neq 0$,

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

Problem 5.

a. (3 points) Calculate $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$.

b. (2 points) Evaluate $\lim_{x\to 1} x^{1/(1-x)}$.

Problem 6. Prove that, of all rectangles of a given perimeter, the square has the largest area.