MAT 132

Practice Final Exam. May 9, 2018

This is a closed notes/ closed book/ electronics off exam.

Please write legibly and show your work.

Each problem is worth 20 points.

Full Name:				
Problem	1	2	3	4
Grade				
Problem	5	6	7	Total
Grade				

Problem 1. Decide whether each series converges or diverges. Justify your answer.

a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\log n}$
b. $\sum_{n=1}^{\infty} \frac{n+1}{e^{\sqrt{n}}}$
c. $\sum_{n=1}^{\infty} \sin\left(\frac{(-1)^n}{n}\right)$
d. $\sum_{n=1}^{\infty} \frac{n-3}{n^3+1}$

Solution 1.

a. This converges by the alternating series test, since $\frac{1}{1+\log x}$ is decreasing and positive on $[1, \infty)$.
b. This converges by comparison with $\sum \frac{1}{n^2}$ since

$$\lim_{n\to\infty} \frac{\frac{n+1}{e^{\sqrt{n}}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(n+1)}{e^{\sqrt{n}}} = \lim_{x\to\infty} \frac{x^4(x^2+1)}{e^x} = 0.$$

c. Write this as

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right).$$

This converges by the alternating series test since $\sin\left(\frac{1}{x}\right)$ is positive and decreasing on $[1, \infty)$.
d. Since $\frac{n-3}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$, and $\frac{n-3}{n^3+1} > 0$ for $n > 3$, the series converges by comparison with $\sum \frac{1}{n^2}$.

Problem 2.

a. Find the Taylor series of $\sqrt{x}$ about $x = 1$ and determine the interval of convergence.
b. Evaluate as an infinite series $\int e^{-t^2} dt$.

Solution 2.

a. Let $y = x - 1$. The Taylor series of $(1+y)^{\frac{1}{2}}$ is given by the binomial series

$$(1+y)^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} y^n$$

where

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\frac{-1}{2}\cdots\frac{3-2n}{2}}{n!}.$$

The radius of convergence is 1, see notes from lecture.

b. Substitute $x = -t^2$ in the Taylor series for e^x to obtain

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}.$$

Since this converges absolutely on $(-\infty, \infty)$

$$\int e^{-t^2} dt = C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)n!}.$$

Problem 3. Approximate $\sin\left(\frac{1}{2}\right)$ correct to within 10^{-6} by performing a Taylor expansion about 0.

Solution 3. The Mclaurin series for $\sin x$ gives

$$\sin\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{2n+1}(2n+1)!}.$$

This is an alternating series with decreasing terms, so the error from taking a partial sum is bounded by the size of the next term in the series. Thus the error in estimating $\sin\left(\frac{1}{2}\right)$ with the degree 7 Taylor polynomial at 0 is bounded by

$$\frac{1}{2^9 9!} = \frac{1}{512 \cdot 9!} < 5.4 \times 10^{-9}.$$

The approximation is $\frac{1}{2} - \frac{1}{2^3 \times 3!} + \frac{1}{2^5 \times 5!} - \frac{1}{2^7 \times 7!} = 0.4794255.$

Problem 4. Solve the following initial value problems.

a. $\frac{dy}{dx} = x(1-y^2)$, $y(0) = 0$.
b. $\frac{dy}{dx} = yx\sqrt{1+x^2}$, $y(0) = 1$.

Solution 4.

a. The equation is separable, so integrating

$$\int \frac{dy}{1-y^2} = \frac{1}{2}\int \left(\frac{1}{1+y} + \frac{1}{1-y}\right) dy = \int x dx$$

Thus

$$\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) = C + \frac{x^2}{2}.$$

Plugging in $y = 0$ obtains $C = 0$. Hence

$$y = \frac{e^{x^2}-1}{e^{x^2}+1}.$$

b. The equation is separable, so integrating

$$\int \frac{dy}{y} = \int x\sqrt{1+x^2}dx$$

In the second integral, substituting $u = 1 + x^2$, $du = 2xdx$ obtains

$$\ln|y| = C + \frac{1}{3}u^{\frac{3}{2}} = C + \frac{1}{3}(1+x^2)^{\frac{3}{2}}.$$

Thus

$$y = Ae^{\frac{1}{3}(1+x^2)^{\frac{3}{2}}}.$$

Plugging in $x = 0$ obtains $A = e^{-\frac{1}{3}}$ so

$$y = e^{\frac{1}{3}((1+x^2)^{\frac{3}{2}}-1)}.$$

Problem 5.

a. Find the length of the curve (e^{2t}, e^{3t}), $0 \le t \le 10$.

b. Find the center of mass of the figure

$$A = \{(x,y) : -5 \le x \le 5, x^2 \le y \le 25\}.$$

Solution 5.

a. $(x(t), y(t)) = (e^{2t}, e^{3t})$, $(x'(t), y'(t)) = (2e^{2t}, 3e^{3t})$. Thus

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{4e^{4t} + 9e^{6t}} = e^{2t}\sqrt{4 + 9e^{2t}}.$$

The arc-length is

$$L = \int_0^{10} e^{2t}\sqrt{4 + 9e^{2t}}dt.$$

Substitute $u = 4 + 9e^{2t}$, $du = 18e^{2t}dt$. Thus

$$L = \frac{1}{18}\int_{13}^{4+9e^{20}} u^{\frac{1}{2}} du = \frac{1}{27}\left[(4 + 9e^{20})^{\frac{3}{2}} - 13\sqrt{13}\right].$$

b. By symmetry, the center of mass of the figure lies on the y axis. The total mass is

$$m = \int_{-5}^{5} (25 - x^2)dx = \left[25x - \frac{x^3}{3}\right]_{-5}^{5} = \frac{500}{3}.$$

The y moment is

$$M_y = \int_{-5}^{5} \frac{1}{2}(25^2 - (x^2)^2)dx = \frac{1}{2}\left[625x - \frac{x^5}{5}\right]_{-5}^{5} = 2500.$$

Thus the center of mass is $(\overline{x}, \overline{y}) = (0, 15)$.

Problem 6. Perform the following indefinite integrals.

a. $\int \frac{x}{1+x^4} dx$
b. $\int x^3 \sin x dx$
c. $\int \frac{dx}{x^3+5x}$
d. $\int \frac{dx}{x(\log x)^2}$.

Solution 6.

a. Substitute $u = x^2$, $du = 2xdx$.

$$\int \frac{x}{1+x^4} dx = \frac{1}{2} \int \frac{du}{1+u^2} = C + \frac{1}{2} \arctan u = C + \frac{1}{2} \arctan x^2.$$

b. Integrate by parts with $u = x^3$, $du = 3x^2 dx$, $dv = \sin x dx$, $v = -\cos x$. Hence

$$\int x^3 \sin x dx = -x^3 \cos x + \int 3x^2 \cos x dx.$$

Integrate by parts again with $u = 3x^2$, $du = 6xdx$, $dv = \cos x dx$, $v = \sin x$. Hence

$$\int x^3 \sin x dx = -x^3 \cos x + 3x^2 \sin x - \int 6x \sin x dx.$$

Integrate by parts again with $u = 6x$, $du = 6dx$, $dv = \sin x dx$, $v = -\cos x$. Hence

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \int \cos x dx \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x. \end{aligned}$$

c. Write $\frac{1}{x^3+5x} = \frac{A}{x} + \frac{Bx+C}{x^2+5}$. Cross multiplying,

$$1 = A(x^2+5) + x(Bx+C).$$

Setting $x = 0$, $A = \frac{1}{5}$. This forces $C = 0$ and $B = -\frac{1}{5}$. Thus

$$\begin{aligned} \int \frac{dx}{x^3+5x} &= \int \frac{dx}{5x} - \int \frac{1}{5} \frac{xdx}{x^2+5} \\ &= \frac{1}{5} \ln|x| - \frac{1}{10} \ln(x^2+5) + C. \end{aligned}$$

d. Substitute $u = \log x$, $du = \frac{dx}{x}$.

$$\int \frac{dx}{x(\log x)^2} = \int \frac{du}{u^2} = -u^{-1} + C = -(\log x)^{-1} + C.$$

Problem 7. The population of wolves (W) and rabbits (R) in an ecosystem is governed by the predator-prey equations

$$\frac{dR}{dt} = 0.1R - 0.002RW, \qquad \frac{dW}{dt} = -0.01W + 0.0002RW.$$

Find any equilibrium points for the system, and derive the family of curves describing the periodic $R - W$ phase trajectories.

Solution 7. Since

$$\begin{aligned} \frac{dR}{dt} &= R(0.1 - 0.002W) \\ \frac{dW}{dt} &= W(-0.01 + 0.0002R). \end{aligned}$$

Equilibria occur where $R = W = 0$ and where $(R, W) = (50, 50)$. By the chain rule,

$$\frac{dW}{dR} = \frac{W}{R}\frac{-0.01 + 0.0002R}{0.1 - 0.002W}.$$

This equation is separable, and is integrated to

$$\int \frac{0.1 - 0.002W}{W} dW = -\int \frac{0.01 - 0.0002R}{R} dR$$

or

$$0.1 \ln W - 0.002W + 0.01 \ln R - 0.0002R = C.$$