MAT 132

Practice Midterm I, Solutions

This is a closed notes/ closed book/ electronics off exam.

Please write legibly and show your work.

Each problem is worth 25 points.

Full Name:

Problem	1	2	3	4	Total
Grade					

Problem 1. Perform the following indefinite integrals.

a.
$$\int \frac{dx}{x^2 + x + 1}$$

b.
$$\int x^2 \cos x dx$$

c.
$$\int \frac{dx}{x^2 + 5x + 6}$$

d.
$$\int \tan x dx$$

Solution 1.

a. Substitute $u = x + \frac{1}{2}$, $du = dx$

$$\begin{aligned}\int \frac{dx}{x^2+x+1} &= \int \frac{du}{u^2+\frac{3}{4}} \\ &= \sqrt{\frac{4}{3}}\tan^{-1}\left(\sqrt{\frac{4}{3}}u\right) + C \\ &= \boxed{\sqrt{\frac{4}{3}}\tan^{-1}\left(\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right)\right) + C}.\end{aligned}$$

b. Integrate by parts twice, first with $u = x^2$, $du = 2xdx$, $dv = \cos xdx$, $v = \sin x$, then with $u = x$, $du = dx$, $dv = \sin xdx$, $v = -\cos x$, to obtain

$$\begin{aligned}\int x^2\cos xdx &= x^2\sin x - 2\int x\sin xdx \\ &= x^2\sin x + 2x\cos x - 2\int \cos xdx \\ &= \boxed{x^2\sin x + 2x\cos x - 2\sin x + C}.\end{aligned}$$

c. By partial fractions $\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$. Thus

$$1 = A(x+3) + B(x+2)$$

so $A = 1$, $B = -1$. Thus

$$\begin{aligned}\int \frac{dx}{x^2+5x+6} &= \int \frac{dx}{x+2} - \int \frac{dx}{x+3} \\ &= \boxed{\log|x+2| - \log|x+3| + C}.\end{aligned}$$

d. Substitute $u = \cos x$, $du = -\sin xdx$, so that

$$\int \tan xdx = -\int \frac{du}{u} = -\log|u| + C = \boxed{-\log|\cos x| + C}.$$

Problem 2. Perform the following definite integrals.

a.

$$\int_0^2 \sqrt{4-x^2}dx.$$

b.

$$\int_0^1 \log x dx$$

c.

$$\int_0^{2\pi} (\sin x)^3 dx$$

d.

$$\int_{-\infty}^{\infty} xe^{-x^2} dx$$

Solution 2.

a. This is the area of a quarter circle of radius 2, hence $\frac{1}{4}\pi\cdot 2^2 = \boxed{\pi}$.

b. This is an improper integral, since $\log x \to -\infty$ as $x \downarrow 0$. Integrate by parts with $u = \log x$, $du = \frac{dx}{x}$, $dv = dx$, $v = x$

$$\begin{aligned}
\int_0^1 \log x dx &= \lim_{t\downarrow 0} \int_t^1 \log x dx \\
&= \lim_{t\downarrow 0} \left([x \log x]_t^1 - \int_t^1 dx \right) \\
&= \lim_{t\downarrow 0} \left[x \log x - x \right]_t^1 .
\end{aligned}$$

Write $t \log t = \frac{\log t}{\frac{1}{t}}$ which is indeterminant of type $\frac{-\infty}{\infty}$ as $t \downarrow 0$. By L'Hospital,

$$\lim_{t\downarrow 0} t \log t = \lim_{t\downarrow 0} \frac{\log t}{\frac{1}{t}} = \lim_{t\downarrow 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\downarrow 0} -t = 0.$$

Hence the limit of the lower evaluation is 0, and the integral is $\boxed{-1}$.

c. By periodicity, $\int_0^{2\pi} \sin^3 x dx = \int_{-\pi}^{\pi} \sin^3 x dx$. Since $\sin^3 x$ is odd, the integral is 0. Alternatively, write $\sin^2 x = 1 - \cos^2 x$, and substitute $u = \cos x$, $du = -\sin x dx$, so that

$$\begin{aligned}
\int_0^{2\pi} \sin^3 x dx &= \int_0^{2\pi} (1 - \cos^2 x) \sin x dx \\
&= -\int_1^1 (1 - u^2) du = \boxed{0}.
\end{aligned}$$

d. Substitute $u = x^2$, $du = 2x dx$ to find

$$\begin{aligned}
\int_0^{\infty} x e^{-x^2} dx &= \lim_{t\to\infty} \int_0^t x e^{-x^2} dx \\
&= \lim_{t\to\infty} \frac{1}{2} \int_0^{t^2} e^{-u} du \\
&= \lim_{t\to\infty} \frac{1}{2} \left[1 - e^{-t^2} \right] = \frac{1}{2}.
\end{aligned}$$

Since the function is odd, the integral over $x < 0$ is the negative, so the indefinite integral exists and is $\boxed{0}$.

Problem 3.

a. Write down the Riemann sum for the integral

$$\int_1^2 \frac{dx}{1+x^2}$$

using left end points.

b. Let $F(x) = \int_{e^x}^{e^{2x}} \frac{dt}{t^2}$. Find $F'(x)$.

Solution 3.

a. With N subdivisions,

$$\Delta x = \frac{2-1}{N} = \frac{1}{N}.$$

The ith left endpoint is

$$x_i = 1 + (i-1)\Delta x = 1 + \frac{i-1}{N}.$$

Hence the Riemann sum is

$$\sum_{i=1}^{N} f(x_i)\Delta x = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{1+x_i^2} = \boxed{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{1+\left(1+\frac{i-1}{N}\right)^2}}.$$

b. Let $G(t)$ be an anti-derivative for $\frac{1}{t^2}$, $G'(t) = \frac{1}{t^2}$, so that by the Evaluation Theorem, $F(x) = G(e^{2x}) - G(e^x)$. By the chain rule

$$\begin{aligned} F'(x) &= G'(e^{2x})(2e^{2x}) - G'(e^x)e^x \\ &= \frac{2e^{2x}}{(e^{2x})^2} - \frac{e^x}{(e^x)^2} \\ &= \boxed{\frac{2}{e^{2x}} - \frac{1}{e^x}}. \end{aligned}$$

Problem 4. At time t the velocity of a particle moving along a line is given by $v(t) = t^3 - t^2 + t - 1$. Find the displacement and distance traveled by the particle between times $t = 0$ and $t = 4$.

Solution 4. The displacement is the integral of velocity,

$$\begin{aligned} s(4) - s(0) &= \int_0^4 (t^3 - t^2 + t - 1)dt \\ &= \left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t\right]_0^4 = \boxed{\frac{140}{3}}. \end{aligned}$$

The distance traveled is the integral of the absolute value of velocity. Factor $t^3 - t^2 + t - 1 = (t-1)(t^2+1)$. This changes from negative to positive at $t = 1$, so the distance traveled is

$$\begin{aligned} d &= \int_0^4 |t^3 - t^2 + t - 1|dt \\ &= \int_0^1 (-t^3 + t^2 - t + 1)dt + \int_1^4 (t^3 - t^2 + t - 1)dt \\ &= \left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t\right]_1^4 - \left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t\right]_0^1 = \boxed{\frac{287}{6}}. \end{aligned}$$