Zeta functions of prehomogeneous vector spaces

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Outline

1. Prehomogeneous vector spaces
2. The classical theory of Sato and Shintani
3. Upgrading the position
   - Local analysis and orbital exponential sums
   - Spectral analysis of the homogeneous spaces
   - Global analysis and subconvexity
4. Ideas in the proofs

Prehomogeneous vector spaces

A prehomogeneous vector space (V, G, ρ) consists of

- A rational representation of a complex algebraic Lie group G
- A singular set S which is a proper algebraic subset of V, such that $V \setminus S$ is a single G orbit
- We assume a real structure, such that $V_{\mathbb{R}} \setminus S$ splits into finitely many open $G_{\mathbb{R}}$ orbits $V_1, ..., V_l$
- An invariant, the discriminant P is a homogeneous polynomial of degree d such that $S = \{x \in V_{\mathbb{R}} : P(x) = 0\}$.
- We assume the representation is rational, and are interested in enumerating integral orbits ordered by discriminant.
- The analytic study of the zeta functions enumerating these orbits was begun by Sato and Shintani

Example spaces with arithmetic applications

The theory of prehomogeneous vector spaces has received significant attention due to applications to arithmetic.

- Delone-Fadeev and Gan-Gross-Savin showed that the prehomogeneous vector space of binary cubic forms $\{f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3\}$ acted on by $\mathrm{GL}_2(\mathbb{R})$, $g \cdot f(x,y) = \frac{f((x,y)g)}{\det g}$ is in natural discriminant preserving bijection with isomorphism classes of cubic rings.
- Davenport and Heilbronn used this bijection to prove an asymptotic count of cubic number fields ordered by discriminant.

Bhargavaology

- Manjul Bhargava proved a discriminant preserving bijection between the space $2 \otimes \mathrm{sym}^2(\mathbb{Z}^3)$ of pairs of integral ternary quadratic forms acted on by $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$ and the space of pairs (Q, C) of quartic rings and a cubic resolvent ring, and used this bijection to count S_4 quartic fields ordered by discriminant, with a similar result in the quintic case.
- Bhargava and Gross have outlined a program using geometric invariant theory, and there have been many recent results counting integral orbits in representation spaces ordered by invariants.

The analytic theory

While the algebraic and geometric theory has seen impressive recent advances there is a long history developing the analytic theory of zeta functions of prehomogeneous vector spaces.

- Sato and Kimura classified prehomogeneous vector spaces in the 60s
- Sato and Shintani proved meromorphic continuation and functional equations of the zeta functions
- Datskovsky, Wright and Yukie developed the adelization of the zeta functions in the 90s, making conjectures regarding number fields

Recent work of myself, Lee, Taniguchi and Thorne among others develops the analytic properties by

- Classifying orbits locally and their orbital exponential sums
- Developing the spectral theory of the underlying homogeneous spaces
- Proving subconvexity of the zeta functions in the critical strip

Outline

The orbital zeta function

- Let G^+ be the connected component of the identity in $G_{\mathbb{R}}$ and $\Gamma = G^+ \cap G_{\mathbb{Z}}$.
- Define a rational character χ of G by $P(g \cdot x) = \chi(g)P(x)$
- Let L be a Γ invariant lattice, and let $L' = L \setminus (L \cap S)$.
- Define the orbital integrals, for Schwarz class f,

$$\Phi_i(f, s) = \int_{V_i} f(x)|P(x)|^s dx.$$

- The orbital zeta function is

$$Z(f, L, s) = \int_{G^+/\Gamma} \chi(g)^s \sum_{x \in L'} f(\rho(g)x) dg$$

The Sato-Shintani zeta functions

By unfolding the orbital integral, obtain the Dirichlet series factorization:

$$\begin{aligned} Z(f, L, s) &= \int_{G^+/\Gamma} \chi(g)^s \sum_{x \in L'} f(\rho(g)x) dg \\ &= \sum_{i=1}^{l} \xi_i(s, L) \Phi_i \left(f, s - \frac{n}{d} \right) \end{aligned}$$

where

$$\xi_i(s, L) = \sum_{x \in \Gamma \backslash L \cap V_i} \frac{1}{|\operatorname{Stab}(x)||P(x)|^s}.$$

Functional equation and meromorphic continuation

Sato and Shintani prove the following general theorem proving a functional equation and meromorphic continuation of the zeta function. Let L^* be the lattice dual to L.

Theorem

The Dirichlet series $\xi_i(s, L), \xi_i^*(s, L^*)$ have meromorphic continuation to $\mathbb{C}$. Moreover, there is a product of Gamma factors $\gamma(s)$, a constant b_0 and trigonometric polynomials $u_{ij}(s)$ such that the functional equation holds

$$v(L^*)\xi_i^*\left(\frac{n}{d} - s, L^*\right) = \gamma\left(s - \frac{n}{d}\right)(2\pi)^{-ds}|b_0|^s \exp\left(\frac{\pi i d s}{2}\right) \times \sum_{j=1}^{l} u_{ji}(s)\xi_j(s, L).$$

The case of binary cubic forms

In the cubic case of binary cubic forms, Shintani made a more detailed study, determining $\gamma(s)$, the trigonometric polynomials, and identifying the poles and residues of the zeta functions at $s = 1$ and $s = \frac{5}{6}$. His analysis was partly carried forward by Yukie in the quartic case, but the residues of the poles are left undetermined.

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Analytic refinements

We discuss analytic refinements of Sato-Shintani's method in three directions.

1. Classify the orbits describing local conditions mod p and mod p^2 and calculate their finite Fourier transforms.
2. Expand G^+/Γ spectrally in automorphic forms. This describes the finer distribution of points in the underlying homogeneous space or space of lattices.
3. Estimate the growth of the zeta functions in vertical strips.

Strong form of the Davenport-Heilbronn Theorem

Taniguchi and Thorne proved the following strong form of the Davenport-Heilbronn Theorem counting cubic fields ordered by discriminant.

Theorem (Taniguchi-Thorne)

Let $N_3^{\pm}(X)$ be the number of cubic fields K with $0 < \pm \mathrm{Disc}(K) < X$. Let $C^- = 3, C^+ = 1, K^- = \sqrt{3}, K^+ = 1$. Then

$$N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)^3\zeta\left(\frac{5}{3}\right)}X^{\frac{5}{6}} + O\left(X^{\frac{7}{9}+\epsilon}\right).$$

Local analysis in the cubic case

A key ingredient in Taniguchi-Thorne's proof is the following local estimate.

- Let $\hat{h}(m)$ be the class number of dual binary cubic forms of discriminant m, with representatives $\{\hat{x}_{i,m}\}_{i=1}^{\hat{h}(m)}$.
- Let $\mathscr{N}_p$ be the indicator function that a form is non-maximal at p, and $\mathscr{N}_q = \prod_{p|q} \mathscr{N}_p$.
- Uniformly in q and Y, for all $\epsilon > 0$,

$$\sum_{0<|m|<Y} \sum_{i=1}^{\hat{h}(m)} |\hat{\mathscr{N}_q}(\hat{x}_{i,m})| \ll_\epsilon q^{-7+\epsilon} Y.$$

Taniguchi-Thorne's classification of mod p quartic orbits

Throughout ℓ denotes a non-square modulo p and $s(a,b,c,d) = (p-1)^a p^b (p+1)^c (p^2+p+1)^{\frac{d}{2}}$.

Orbit	Representative	Orbit size	Stabilizer size
$\mathscr{O}_0$	$(0,0)$	1	$s(5,4,2,2)$
$\mathscr{O}_{D1^2}$	$(0,w^2)$	$s(1,0,1,2)$	$s(4,4,1,0)$
$\mathscr{O}_{D11}$	$(0,vw)$	$s(1,1,2,2)/2$	$2s(4,3,0,0)$
$\mathscr{O}_{D2}$	$(0,v^2-\ell w^2)$	$s(2,1,1,2)/2$	$2s(3,3,1,0)$
$\mathscr{O}_{Dns}$	$(0,u^2-vw)$	$s(2,2,1,2)$	$s(3,2,1,0)$
$\mathscr{O}_{Cs}$	(w^2,vw)	$s(2,1,2,2)$	$s(3,3,0,0)$
$\mathscr{O}_{Cns}$	(vw,uw)	$s(2,3,1,2)$	$s(3,1,1,0)$
$\mathscr{O}_{B11}$	(w^2,v^2)	$s(2,2,2,2)/2$	$2s(3,2,0,0)$
$\mathscr{O}_{B2}$	$(vw,v^2+\ell w^2)$	$s(3,2,1,2)/2$	$2s(2,2,1,0)$
$\mathscr{O}_{1^4}$	$(w^2,uw+v^2)$	$s(3,2,2,2)$	$s(2,2,0,0)$
$\mathscr{O}_{1^31}$	$(vw,uw+v^2)$	$s(3,3,2,2)$	$s(2,1,0,0)$
$\mathscr{O}_{1^21^2}$	(w^2,uv)	$s(2,4,2,2)/2$	$2s(3,0,0,0)$
$\mathscr{O}_{2^2}$	$(w^2,u^2-\ell v^2)$	$s(3,4,1,2)/2$	$2s(2,0,1,0)$
$\mathscr{O}_{1^211}$	(v^2-w^2,uw)	$s(3,4,2,2)/2$	$2s(2,0,0,0)$
$\mathscr{O}_{1^22}$	$(v^2-\ell w^2,uw)$	$s(3,4,2,2)/2$	$2s(2,0,0,0)$
$\mathscr{O}_{1111}$	$(uw-vw,uv-vw)$	$s(4,4,2,2)/24$	$24s(1,0,0,0)$
$\mathscr{O}_{112}$	$(vw,u^2-v^2-\ell w^2)$	$s(4,4,2,2)/4$	$4s(1,0,0,0)$
$\mathscr{O}_{22}$	$(vw,u^2-\ell v^2-\ell w^2)$	$s(4,4,2,2)/8$	$8s(1,0,0,0)$
$\mathscr{O}_{13}$	$(uw-v^2,B_3)$	$s(4,4,2,2)/3$	$3s(1,0,0,0)$
$\mathscr{O}_4$	$(uw-v^2,B_4)$	$s(4,4,2,2)/4$	$4s(1,0,0,0)$

(1)

The items B_3 and B_4 indicate $B_3 = uv + a_3v^2 + b_3vw + c_3w^2$ and $B_4 = u^2 + a_4uv + b_4v^2 + c_4vw + d_4w^2$ where $X^3 + a_3X^2 + b_3X + c_3$ and $X^4 + a_4X^3 + b_4X^2 + c_4X + d_4$ are irreducible over $\mathbb{Z}/p\mathbb{Z}$.

Local analysis in the quartic case

Let $\mathbf{1}_{\text{non-max}}$ be the indicator function of forms in $V(\mathbb{Z}/p^2\mathbb{Z})$ which correspond to quartic rings non-maximal at p.

Theorem (H., 2020)

For $p > 3$ the Fourier transform

$$\widehat{\mathbf{1}_{\text{non-max}}}(\xi) = \sum_{x \in V(\mathbb{Z}/p^2\mathbb{Z})} \mathbf{1}_{\text{non-max}}(x) e_{p^2}([x, \xi])$$

is supported on the mod p *orbits $\mathscr{O}_0$, $\mathscr{O}_{D1^2}$, $\mathscr{O}_{D11}$ and $\mathscr{O}_{D^2}$. It satisfies*

$$\begin{aligned}\left\|\widehat{\mathbf{1}_{\text{non-max}}}\right\|_1 &= 2p^{29} + 2p^{28} + 4p^{27} - 8p^{26} - 19p^{25} - 2p^{24} + 20p^{23} + 24p^{22} - 5p^{21} \\ &\quad - 17p^{20} - 5p^{19} + 3p^{18} + 2p^{17} - 2p^{16} + p^{15} + p^{14}, \\ \left\|\widehat{\mathbf{1}_{\text{non-max}}}\right\|_2^2 &= p^{46} + 2p^{45} + 2p^{44} - 3p^{43} - 4p^{42} - p^{41} + 3p^{40} + 3p^{39} - p^{38} - p^{37}, \\ \left|\operatorname{supp} \widehat{\mathbf{1}_{\text{non-max}}}\right| &= 2p^{15} + p^{14} - 2p^{13} - p^{12} + 2p^{10} - p^{8}.\end{aligned}$$

An exact formula for the Fourier transform is given along with a classification of mod p^2 orbits.

Definition of the automorphic twist

The automorphic twist of a Sato-Shintani zeta function is defined as follows.

Let ϕ be an automorphic form for $\Gamma\backslash G^1$.

For each open orbit V_i choose x_i a base point, $|\mathrm{Disc}(x_i)| = 1$. The twisted zeta function is

$$\xi_i(\phi, s, L) = \sum_{x \in \Gamma\backslash L \cap V_i} \sum_{g \cdot x_i = x} \frac{\phi(g)}{|\mathrm{Stab}(x)||\mathrm{Disc}(x)|^s}.$$

The convolution equation

Let $f \in C_c^\infty(G^1)$ be bi-K-invariant.
For imaginary $z = i\gamma$, $E(i\gamma, g) = \mathbf{E}_{\frac{1+\gamma^2}{4}}(g^t)$, which is left invariant under Γ.
As a right convolution operator f acts on $\mathbf{E}_r$ as multiplication by a scalar. To check this, note that

$$\mathbf{E}_r * f(g_0) = \int_{gh=g_0} \mathbf{E}_r(g) f(h) dh \tag{2}$$

is left Γ invariant and right K invariant. Also, it is an eigenfunction of the Laplacian and Hecke operators, with the same eigenvalues as $\mathbf{E}_r$. It follows by multiplicity one that the convolution is a multiple of $\mathbf{E}_r$.

The convolution equation

The following lemma determines the eigenvalue.

Lemma

We have

$$\mathbf{E}_r * f = \left(\int_{G^1} f(g) t(g)^{1+z} dg \right) \mathbf{E}_r.$$

Determination of eigenvalue

Proof.

Let $\psi \in C_c^\infty(\Gamma\backslash G^1/K)$ be a smooth test function and let ψ_0 be the constant term in its Fourier expansion in the parabolic direction. The Petersson inner product of $\mathbf{E}_r$ with ψ is a Mellin transform of ψ_0,

$$\begin{aligned}\int_{\Gamma\backslash G^1} \mathbf{E}_r(g)\overline{\psi(g)}dg &= \int_{\Gamma\backslash G^1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} t(\gamma g)^{1+z}\overline{\psi(\gamma g)}dg \\ &= \int_{\Gamma_\infty\backslash G^1} t(g)^{1+z}\overline{\psi(g)}dg \\ &= \int_0^\infty \psi_0(t)t^{-1+z}\frac{dt}{t} = \tilde{\psi}_0(-1+z).\end{aligned}$$

□

Determination of eigenvalue

Proof.

Next we calculate the inner product with the convolution $\mathbf{E}_r * f$,

$$\begin{aligned}\int_{\Gamma\backslash G^1} (\mathbf{E}_r * f)(g)\overline{\psi(g)}dg &= \int_{\Gamma\backslash G^1}\int_{G^1} \mathbf{E}_r(h)f(h^{-1}g)\overline{\psi(g)}dhdg \\ &= \int_{\Gamma\backslash G^1}\int_{G^1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} t(h)^{1+z}f(h^{-1}\gamma g)\overline{\psi(\gamma g)}dhdg \\ &= \int_{\Gamma_\infty\backslash G^1}\int_{G^1} t(h)^{1+z}f(h^{-1}g)\overline{\psi(g)}dhdg \\ &= \int_0^\infty \frac{dt_1}{t_1^3}\int_0^\infty \frac{dt_2}{t_2^3}\int_{-\infty}^\infty du t_1^{1+z} f\left(\begin{pmatrix} \frac{t_2}{t_1} & \frac{u}{t_1 t_2} \\ 0 & \frac{t_1}{t_2}\end{pmatrix}\right)\psi_0(t_2).\end{aligned}$$

After a change of coordinates we obtain

$$\tilde{\psi}_0(-1+z)\int_0^\infty \frac{dt}{t}t^z\int_{-\infty}^\infty duf\left(\begin{pmatrix}\frac{1}{t} & u \\ 0 & t\end{pmatrix}\right) = \tilde{\psi}_0(-1+z)\int_0^\infty \frac{dt}{t}t^{1+z}\int_{-\infty}^\infty duf\left(n_u a_t\right).$$

□

The orbital zeta function with an automorphic twist

Let f_G be defined on G^1 by $f_G(g) = \exp(-\operatorname{tr} g^t g)$ and extend f_G to G^+ independent of the determinant.
Let $f_D(x) \in C_c^\infty(\mathbb{R}^+)$. Define

$$f_\pm(g \cdot x_\pm) = f_G(g) f_D(\chi(g)). \tag{3}$$

The twisted orbital integrals are given by

$$Z^\pm(f_\pm, \mathbf{E}_r, L; s) = \int_{G^+/\Gamma} \chi(g)^s \mathbf{E}_r(g^{-1}) \sum_{x \in L} f_\pm(g \cdot x) dg. \tag{4}$$

Lemma

In $\operatorname{Re}(s) > 1$,

$$Z^\pm(f_\pm, \mathbf{E}_r, L; s) = \frac{\sqrt{\pi} K_{\frac{z}{2}}(2)}{12} \mathscr{L}^\pm(\mathbf{E}_r, s) \tilde{f}_D(s). \tag{5}$$

The shape of a number field

- A degree n number field $K/\mathbb{Q}$ has r_1 real and r_2 complex embeddings, $n = r_1 + 2r_2$.
- Let the real embeddings be $\sigma_1, ..., \sigma_r : K \to \mathbb{R}$ and the complex embeddings be $\sigma_{r_1+1}, ..., \sigma_{r_1+r_2} : K \to \mathbb{C}$.
- The *canonical embedding* is

$$\sigma(x) = (\sigma_1(x), ..., \sigma_{r_1+r_2}(x)) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

The shape of a number field

- Treat $\mathbb{C}$ as a two-dimensional vector space over $\mathbb{R}$. The ring of integers $\mathscr{O} \subset K$ is a n-dimensional lattice in $\mathbb{R}^n$ under the mapping

$$x \mapsto (\sigma_1(x), ..., \sigma_{r_1}(x), ..., \mathrm{Re}\,\sigma_{r_1+r_2}(x), \mathrm{Im}\,\sigma_{r_1+r_2}(x)).$$

- The covolume of this lattice is $\mathrm{vol}(\sigma(\mathscr{O})) = 2^{-r_2}|D|^{\frac{1}{2}}$ where $D = D_K$ is the field discriminant.

The shape of a number field

- An old theorem of Hermite states that there are only finitely many number fields of a given discriminant.
- Note that, since $\sigma(1)$ is always present in the ring of integers, compared to the volume this is a short vector in the lattice.
- Thus the lattice shape Λ_K is understood to be the $(n-1)$-dimensional orthogonal projection in the space orthogonal to $\sigma(1)$, rescaled to have volume 1.

Past work

- Terr in the case $n = 3$ and Bhargava and Harron in the cases $n = 3, 4, 5$ show that when $\mathfrak{S}_n$ fields are ordered by discriminant, the field shape Λ_K becomes asymptotically equidistributed in the space

$$\mathscr{S}_{n-1} := \mathrm{GL}_{n-1}(\mathbb{Z})\backslash GL_{n-1}(\mathbb{R})/\mathrm{GO}_{n-1}(\mathbb{R}).$$

- This means that for $f \in C_c(\mathscr{S}_{n-1})$,

$$\lim_{X\to\infty} \sum_{\substack{[K:\mathbb{Q}]=n,\mathfrak{S}_n \\ |\operatorname{Disc}(K)|\leq X}} f(\Lambda_K) \Bigg/ \sum_{\substack{[K:\mathbb{Q}]=n,\mathfrak{S}_n \\ |\operatorname{Disc}(K)|\leq X}} 1 = \int_{\mathscr{S}_{n-1}} f(x) d\mu_{n-1}(x)$$

where $d\mu_{n-1}(x)$ is the induced probability Haar measure on $\mathscr{S}_{n-1}$.

- The arguments permit restricting to fields having a prescribed number of complex embeddings.

Automorphic forms

- In the case $n = 3$, a natural basis of functions on the larger quotient

$$\Lambda_2 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$$

  consist of automorphic forms ϕ which transform under $\mathrm{SO}_2(\mathbb{R})$ on the right by a character, and which are joint eigenfunctions of the Casimir operator and the Hecke operators.
- This decomposes Λ_2 spectrally into the constant function, cusp forms (discrete spectrum) and Eisenstein series (continuous spectrum).

Estimate of Weyl sums

Theorem (H., 2019)

Let ϕ be a Hecke-eigen-cusp form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ of K-type $2k$. Let $F \in C_c^\infty(\mathbb{R}^+)$ be a smooth test function. For any $\epsilon > 0$, as $X \to \infty$,

$$N_{3,\pm}(\phi, F, X) := \sum_{[K:\mathbb{Q}]=3} \phi(\Lambda_K) F\left(\frac{\pm\,\mathrm{Disc}(K)}{X}\right) \ll_\phi X^{\frac{2}{3}+\epsilon}.$$

The bound should be compared to the number of cubic fields of discriminant of size at most X, which is order X.

Residues of the Eisenstein twist

Associated to real analytic Eisenstein series $\mathbf{E}_r$, $r = \frac{1+z^2}{4}$ are

$$\mathscr{L}^{\pm}(\mathbf{E}_r, s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{i=1}^{h(\pm m)} \frac{\mathbf{E}_r(g_{i,\pm m})}{|\operatorname{Stab}(x_{i,\pm m})|}, \qquad \operatorname{Re}(s) > \frac{5}{4}.$$

Theorem (H.-Lee, 2021)

The functions $\mathscr{L}^{\pm}(\mathbf{E}_r, s)$ have meromorphic continuation to $\mathbb{C}$, with poles at $\frac{5\pm z}{4}$ and $\frac{11\pm z}{12}$ with residues listed in the following table.

Pole	$\frac{11+z}{12}$	$\frac{5+z}{4}$
$\mathscr{L}^-$	$\frac{\zeta\left(\frac{1-z}{3}\right)2^{\frac{z-1}{6}}\pi^{\frac{2z+1}{6}}}{3} \cos\left(\frac{\pi(1-z)}{6}\right) \frac{\Gamma\left(\frac{1-z}{3}\right)\Gamma\left(\frac{4-z}{6}\right)}{\Gamma\left(\frac{7-z}{6}\right)}$	$\zeta(3+z)2^{\frac{-5-z}{2}}$
$\mathscr{L}^+$	$\frac{\zeta\left(\frac{1-z}{3}\right)2^{\frac{z-1}{6}}\pi^{\frac{2z+1}{6}}}{3^{\frac{7-z}{4}}} \cos\left(\frac{\pi(1-z)}{6}\right) \frac{\Gamma\left(\frac{1-z}{3}\right)\Gamma\left(\frac{4-z}{6}\right)}{\Gamma\left(\frac{7-z}{6}\right)}$	$\zeta(3+z)2^{\frac{-5-z}{2}}3^{\frac{1+z}{4}}$

The poles at $\frac{11-z}{12}$ and $\frac{5-z}{4}$ are found by replacing z with $-z$ and multiplying by $\frac{\xi(z)}{\xi(1+z)}$.

Reducible forms

- Shintani shows that all *reducible* binary cubic forms are $\mathrm{SL}_2(\mathbb{Z})$ equivalent to a form $bx^2y + cxy^2 + dy^3$.
- He identifies this space with the space of *binary quadratic forms* acted on by the Borel subgroup $B = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.
- Modifying Shintani's method, we show that when summation in the Eisenstein series twisted Shintani zeta function is restricted to irreducible forms, the function has meromorphic continuation to $\mathrm{Re}(s) > \frac{3}{4}$, with poles at $\frac{11 \pm z}{12}$, with the residues in the table.

Subconvexity of Shintani's zeta function

Let $V_{\mathbb{Z}} = \{f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 : a,b,c,d \in \mathbb{Z}\}$ be the space of integral binary cubic forms. Shintani introduced zeta functions

$$\xi^{\pm}(s) := \sum_{\substack{f \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}} \\ \pm \mathrm{Disc}(f) > 0}} \frac{1}{|\mathrm{Stab}(f)|} \frac{1}{|\mathrm{Disc}(f)|^s}, \qquad \mathrm{Re}(s) > 1. \tag{6}$$

Given a Maass cusp form ϕ for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$, the twisted version is

$$\mathscr{L}^{\pm}(\phi, s) := \sum_{\substack{f \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}} \\ \pm \mathrm{Disc}(f) > 0}} \frac{\phi(f)}{|\mathrm{Stab}(f)|} \frac{1}{|\mathrm{Disc}(f)|^s}, \qquad \mathrm{Re}(s) > 1. \tag{7}$$

Subconvexity of Shintani's zeta function

- Define the diagonalization

$$\xi^{\mathrm{add}}(s) = 3^{\frac{1}{2}}\xi^{+}(s) + \xi^{-}(s), \qquad \xi^{\mathrm{sub}}(s) = 3^{\frac{1}{2}}\xi^{+}(s) - \xi^{-}(s) \tag{8}$$

and completed zeta functions

$$\begin{aligned}\Lambda^{\mathrm{add}}(s) &= \left(\frac{432}{\pi^4}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+\frac{1}{6}}{2}\right)\Gamma\left(\frac{s-\frac{1}{6}}{2}\right)\xi^{\mathrm{add}}(s), \\ \Lambda^{\mathrm{sub}}(s) &= \left(\frac{432}{\pi^4}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+\frac{5}{6}}{2}\right)\Gamma\left(\frac{s+\frac{7}{6}}{2}\right)\xi^{\mathrm{sub}}(s),\end{aligned} \tag{9}$$

- Self-dual functional equations $\Lambda(s) = \Lambda(1-s)$.
- Degree 4, with analytic conductor $\mathcal{C}(\frac{1}{2}+i\tau) = \tau^4$ as $\tau \to \infty$.
- No Riemann Hypothesis, but it may be conjectured that Lindelöf Hypothesis holds, $\xi^{\mathrm{add}}\left(\frac{1}{2}+i\tau\right), \xi^{\mathrm{sub}}\left(\frac{1}{2}+i\tau\right) \ll_\epsilon \mathcal{C}^\epsilon$.

Subconvexity of Shintani's zeta function

Theorem (H.-Lee, 2021)

The Shintani zeta functions satisfy the sub-convex bound, for any $\epsilon > 0$,

$$\xi^{\mathsf{add}}\left(\frac{1}{2}+i\tau\right), \xi^{\mathsf{sub}}\left(\frac{1}{2}+i\tau\right) \ll_\epsilon \tau^{\frac{98}{99}+\epsilon} \tag{10}$$

as $\tau \to \infty$.

Subconvexity of Shintani's zeta function

Although there is not a known functional equation for the twisted versions, we in fact still prove a subconvex bound.

Theorem (H.-Lee, 2021)

The twisted Shintani zeta functions satisfy the subconvex bound, for any $\epsilon > 0$,

$$\mathscr{L}^{\pm}\left(\frac{1}{2} + i\tau, \phi\right) \ll_{\epsilon,\phi} \tau^{\frac{26}{27}+\epsilon} \tag{11}$$

as $\tau \to \infty$.

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The approximate functional equation

Theorem

Let $G(u)$ be any function which is holomorphic and bounded in $|\operatorname{Re}(u)| < 4$, even, with $G(0) = 1$. For $0 < \operatorname{Re}(s) < 1$,

$$\xi^{\text{add}}(s) = \sum_n \frac{a^{\text{add}}(n)}{n^s} V_s^{\text{add}}\left(\frac{n}{\sqrt{432}}\right) + \epsilon^{\text{add}}(s) \sum_n \frac{a^{\text{add}}(n)}{n^{1-s}} V_{1-s}^{\text{add}}\left(\frac{n}{\sqrt{432}}\right) + R^{\text{add}}(s) \quad (12)$$

$$\xi^{\text{sub}}(s) = \sum_n \frac{a^{\text{sub}}(n)}{n^s} V_s^{\text{sub}}\left(\frac{n}{\sqrt{432}}\right) + \epsilon^{\text{sub}}(s) \sum_n \frac{a^{\text{sub}}(n)}{n^{1-s}} V_{1-s}^{\text{sub}}\left(\frac{n}{\sqrt{432}}\right) + R^{\text{sub}}(s)$$

where $\epsilon^*(s) = 432^{\frac{1}{2}-s} \frac{\gamma^*(1-s)}{\gamma^*(s)}$,

$$V_s^*(y) = \frac{1}{2\pi i} \int_{\operatorname{Re} u = 3} y^{-u} G(u) (432)^{\frac{u}{2}} \frac{\gamma^*(s+u)}{\gamma^*(s)} \frac{du}{u} \quad (13)$$

$$R^{\text{add}}(s) = \left(\operatorname{Res}_{u=1-s} + \operatorname{Res}_{u=\frac{5}{6}-s} + \operatorname{Res}_{u=\frac{1}{6}-s} + \operatorname{Res}_{u=-s}\right) \frac{\Lambda^{\text{add}}(s+u)}{432^{\frac{s}{2}} \gamma^{\text{add}}(s)} \frac{G(u)}{u} \quad (14)$$

$$R^{\text{sub}}(s) = \left(\operatorname{Res}_{u=1-s} + \operatorname{Res}_{u=-s}\right) \frac{\Lambda^{\text{sub}}(s+u)}{432^{\frac{s}{2}} \gamma^{\text{sub}}(s)} \frac{G(u)}{u}.$$

The following lemma estimates the dependence in switching between rectangular and homogeneous coordinates.

Lemma

When u, t, θ vary in a Siegel set and $\lambda \geq 1$, and $v \in B$ the change of coordinates $(a, b, c, d) = n_u a_t k_\theta d_\lambda \cdot v$ satisfies

$$\frac{\partial(a,b,c,d)}{\partial(u,t,\theta,\lambda)} = \begin{pmatrix} 0 & O(\lambda t^{-3}) & O(\lambda t^{-1}) & O(\lambda t) \\ O(\lambda t^{-4}) & O(\lambda t^{-2}) & O(\lambda) & O(\lambda t^2) \\ O(\lambda t^{-3}) & O(\lambda t^{-1}) & O(\lambda t) & O(\lambda t^3) \\ O(t^{-3}) & O(t^{-1}) & O(t) & O(t^3) \end{pmatrix} \tag{15}$$

$$\frac{\partial(u,t,\theta,\lambda)}{\partial(a,b,c,d)} = \begin{pmatrix} O(\lambda^{-1}t^5) & O(\lambda^{-1}t^4) & O(\lambda^{-1}t^3) & O(t^3) \\ O(\lambda^{-1}t^3) & O(\lambda^{-1}t^2) & O(\lambda^{-1}t) & O(t) \\ O(\lambda^{-1}t) & O(\lambda^{-1}) & O(\lambda^{-1}t^{-1}) & O(t^{-1}) \\ O(\lambda^{-1}t^{-1}) & O(\lambda^{-1}t^{-2}) & O(\lambda^{-1}t^{-3}) & O(t^{-3}) \end{pmatrix}.$$

Averaging in cartesian boxes

After applying the approximate functional equation and adjusting the smooth weight, reduce to estimating sums of type

$$\Sigma_1''(Y) = \frac{1}{n_\pm M_\pm} \sum_f{}' \sigma\left(\frac{|\operatorname{Disc}(f)|}{Y}\right) \frac{V_{\frac{1}{2}+i\tau}\left(\frac{|\operatorname{Disc}(f)|}{\sqrt{432}}\right) W(f)}{|\operatorname{Disc}(f)|^{\frac{1}{2}}} \times \mathbf{E}_{y \in B_{R(Y)}}\left[|\operatorname{Disc}(f+y)|^{-i\tau}\right].$$

Here $W(f)$ restricts the form f to lie in a fundamental domain for $\Gamma \backslash G^+ \cdot v$, as in Bhargava's methods of counting forms.
The expectation in y is estimated by Taylor expanding the phase, and applying van der Corput's inequality twice.

Thank you

Thanks for listening!

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