EFFECTIVE ZERO-CYCLES AND THE BLOCH-BEILINSON FILTRATION

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ABSTRACT. A conjecture of Voisin states that given points x and y on a smooth projective variety X whose algebra of holomorphic forms is generated in degree at most 2,

$$x = y \in \mathrm{CH}_0(X) \iff x - y \in F^3_{BB}\mathrm{CH}_0(X).$$

In other words, to ascertain that two points are rationally equivalent to each other one does not need to look very deep in the Bloch-Beilinson filtration. In this note, we suggest a generalization that allows for generators of $H^0(X, \Omega^\bullet)$ in higher degrees and for rational equivalence of effective zero-cycles of higher degree, both at the expense of looking deeper in the Bloch-Beilinson filtration. In the first half, we provide some evidence in support of this conjecture by adapting several of the examples of Voisin. In the second half, in an effort to explain our main conjecture, we formulate a second conjecture predicting when the diagonal of a smooth projective variety X belongs to the subalgebra of $\mathrm{CH}^\bullet(X \times X)$ generated in degree at most d. We explain the relation of this second conjecture with the first and with well known conjectures on algebraic cycles.

1. Introduction

In [Voi22], inspired by recent work of Marian and Zhao [MZ20], Voisin formulates the following conjecture:

Conjecture 1.1 ([Voi22] Conjecture 2.11). *Let X be a smooth projective variety whose algebra of holomorphic forms is generated in degree ≤ 2. Then, given $x, y \in X$,*

$$x = y \in \mathrm{CH}_0(X) \iff x = y \in \mathrm{CH}_0(X)/F^3_{BB}\mathrm{CH}_0(X). \tag{1.1}$$

We work throughout with Chow rings with rational coefficients and $F^\bullet_{BB}$ denotes the candidate for the Bloch-Beilinson filtration which was proposed in [Voi04], namely

$$F^i_{BB}\mathrm{CH}_0(X) = \bigcap_{\Gamma, Y} \ker(\ \Gamma_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(Y)\ ),$$

where Y ranges over all smooth projective $(i-1)$-folds and Γ over all correspondences in $\mathrm{CH}^{i-1}(X \times Y)$.

We propose to generalize Conjecture 1.1 in two directions. The first consists in allowing the algebra of holomorphic forms on X to be generated in higher degrees at the expense of requiring the equality modulo a deeper piece of the Bloch-Beilinson filtration on the right hand side of (1). The second and more interesting direction allows for rational equivalence of effective zero-cycles of higher degree, again at the expense of looking deeper into the Bloch-Beilinson filtration.

Conjecture 1.2. *Let X be a smooth projective variety whose algebra of holomorphic forms is generated in degree $\leq d$. Then, for $x_1, \cdots, x_m, y_1, \cdots, y_m \in X$,*

$$\sum_{i=1}^m x_i = \sum_{i=1}^m y_i \in \mathrm{CH}_0(X) \iff \sum_{i=1}^m x_i = \sum_{i=1}^m y_i \in \mathrm{CH}_0(X)/F^{md+1}_{BB}\mathrm{CH}_0(X). \tag{1.2}$$

Equivalently, the image of the following map intersects $F^{md+1}\mathrm{CH}_0(X)$ trivially:

$$\Phi_m : \qquad \mathrm{Sym}^m X \times \mathrm{Sym}^m X \qquad \longrightarrow \qquad \mathrm{CH}_0(X)$$

$$(x_1 + \cdots + x_m, y_1 + \cdots + y_m) \longmapsto \sum_{i=1}^m x_i - \sum_{i=1}^m y_i.$$

Remark 1.3. Since $F^r_{BB}\mathrm{CH}_0(X) = 0$ for $r \geq \dim X$, Conjecture 1.2 is only interesting for small m and l. For instance, it holds trivially for curves or varieties with an indecomposable top form e.g., Calabi-Yau varieties, complete intersections of general type, etc.

Voisin provides several pieces of evidence in favor of Conjecture 1.1 and we generalize some of these results to the setting of Conjecture 1.2. For abelian varieties we get:

Theorem 2.3. *Let A be an abelian g-fold and let $x_1, \cdots, x_m, y_1, \cdots, y_m \in A$,*

$$\sum_{i=1}^m x_i = \sum_{i=1}^m y_i \ \in \ \mathrm{CH}_0(A) \iff \sum_{i=1}^m x_i - \sum_{i=1}^m y_i \ \in \ \bigoplus_{s=m+1}^{g} \mathrm{CH}^g_{(s)}(A). \tag{1.3}$$

In (1.3) we write $\mathrm{CH}^g_{(s)}(A)$ for the associated graded pieces of the Beauville filtration on $\mathrm{CH}_0(A)$:

$$\mathrm{CH}^g_{(s)}(A) \ =_{\mathrm{def}} \ \{\alpha \in \mathrm{CH}^g(A) : [k]^*(\alpha) = k^{2g-s}\alpha \text{ for all } k \in \mathbb{Z}\},$$

where $[k] : A \longrightarrow A$ is the multiplication by k isogeny. A conjecture of Beauville [Bea86] states that $\mathrm{CH}^g_{(s)}(A) = 0$ for $s < 0$, so Theorem 1.3 is expected to imply Conjecture 1.2 for abelian varieties.

In Proposition 2.6 we adapt an argument of Voisin from [Voi22] to deduce Conjecture 1.2 for hyper-Kähler varieties from well-known conjectures on algebraic cycles.

In the second half of the paper, in an attempt to explain and motivate Conjecture 1.2 we introduce the notion of polynomial decomposition of the diagonal.

Definition 3.2. *A smooth projective n-fold X admits a rational degree l polynomial decomposition of the diagonal if*

$$\Delta_X = Z_1 + Z_2 \in \mathrm{CH}^n(X \times X),$$

where Z_1 belongs to the subalgebra of $\mathrm{CH}^\bullet(X \times X)$ generated in degree $\leq l$ and Z_2 is supported on $D \times X$ for some divisor $D \subset X$.

We show that if X has a rational degree l polynomial decomposition of the diagonal and satisfies Conjecture 2.1 formulated by Nori, then for $x_1, \cdots, x_m, y_1, \cdots, y_m \in X$,

$$\sum_{i=1}^m x_i \ = \ \sum_{i=1}^m y_i \ \in \ \mathrm{CH}_0(X) \quad \iff \quad \sum_{i=1}^m x_i = \sum_{i=1}^m y_i \ \in \ \mathrm{CH}_0(X)/F^{md+1}_{BB}\mathrm{CH}_0(X).$$

Accordingly, the following conjecture in conjunction with Conjecture 2.1 implies Conjecture 1.2:

Conjecture 3.6. *Let X be a smooth projective n-fold. Then X admits a degree l polynomial decomposition of the diagonal if and only if the algebra $H^0(X, \Omega^\bullet)$ is generated in degree $\leq l$.*

Finally, we will show in Proposition 3.9 how Conjecture 3.6 easily implies the generalized Bloch conjecture in coniveau 1.

2. Examples

In this section we present some cases in which Conjecture 1.2 is satisfied either unconditionally or conditional on well known conjectures on algebraic cycles.

2.1. **Abelian varieties.** In [Voi22], Voisin shows that given an abelian variety A, a desingularization of the quotient $A/\pm$ satisfies Conjecture 1.1. This is essentially equivalent to the fact that abelian varieties satisfy Conjecture 1.2 for $m = 2$, and the proof can be adapted to show that the following conjecture of Nori implies Conjecture 1.2 for abelian varieties.

Conjecture 2.1 ([Nor93])**.** *Let X be a smooth projective variety and $w \in \mathrm{CH}^i(X)$ a cycle such that $w|_T = 0 \in \mathrm{CH}_0(T)$ for any i-fold $T \subset X$. Then $w = 0 \in \mathrm{CH}^i(X)$.*

Although we will usually denote by x both a point of X and the corresponding effective zero-cycle in $\mathrm{CH}_0(X)$, when there is a risk of confusion we will use $\{x\}$ to denote the element of $\mathrm{CH}_0(X)$. In particular, if X is an abelian variety, $\{x + y\}$ will denote the effective zero-cycle of degree one obtained by adding x and y in X, whereas $\{x\} + \{y\}$ denotes an effective zero-cycle of degree 2.

Proposition 2.2. *Conjecture 1.2 holds for abelian varieties and $m = 2$. Moreover, Conjecture 2.1 for an abelian g-fold A implies Conjecture 1.2 for A.*

Proof. Suppose

$$\{a\} + \{a'\} = \{b\} + \{b'\} \in \mathrm{CH}_0(A)/F^3_{BB}\mathrm{CH}_0(A).$$

Then $a + a' = b + b'$ so, translating by $-a - a'$ if needed, we can assume that $a' = -a$, $b' = -b$. Voisin shows in Proposition 2.17 of [Voi22] that

$$\{a\} + \{-a\} = \{b\} + \{-b\} \in \mathrm{CH}_0(A)/F^3_{BB}\mathrm{CH}_0(A) \iff \{a\} + \{-a\} = \{b\} + \{-b\} \in \mathrm{CH}_0(A).$$

For the general case we can follow a similar argument. The main differences are the use of the fundamental theorem on symmetric polynomials and the substitution of Conjecture 2.1 in place of a theorem of Joshi [Jos95]. Suppose that

$$\sum_{i=1}^{m} \{x_i\} = \sum_{i=1}^{m} \{y_i\} \in \mathrm{CH}_0(A)/F^{m+1}_{BB}\mathrm{CH}_0(A). \tag{2.1}$$

Let Θ be an ample divisor that gives an isogeny

$$\begin{array}{rcl} A & \longrightarrow & \widehat{A} \\ x & \longmapsto & D_x =_{\mathrm{def}} \Theta_x - \Theta. \end{array}$$

The map $x \mapsto D^j_x \in \mathrm{CH}^j(A)$ is a given by a correspondence in $\mathrm{CH}^j(A \times A)$. Hence, assuming Conjecture 2.1, we can use Lemma 2.5 and (2.1) to deduce that

$$\sum_{i=1}^{m} D^j_{x_i} = \sum_{i=1}^{m} D^j_{y_i}, \qquad \forall j \in \{1, \cdots, m\}.$$

The fundamental theorem on symmetric polynomials then implies that

$$\sum_{i=1}^{m} D^j_{x_i} = \sum_{i=1}^{m} D^j_{y_i} \in \mathrm{CH}^j(A), \qquad \forall j \in \mathbb{N}.$$

The Chow ring of A has two ring structures, one given by intersection and the other by the Pontryagin product:

$$\begin{array}{rcl} \mathrm{CH}^\bullet(A) \times \mathrm{CH}^\bullet(A) & \longrightarrow & \mathrm{CH}^\bullet(A) \\ (\alpha, \beta) & \longmapsto & \sigma_*(\alpha \times \beta), \end{array}$$

where $\sigma : A \times A \longrightarrow A$ is the summation map. A formula of Beauville then gives

$$\sum_{i=1}^{m} \frac{\Theta^{g-j}}{(g-j)!} D_{x_i}^{j} \;=\; \sum_{i=1}^{m} \frac{\Theta^{g}}{g!} * \gamma(x_i)^{*j} \;\in\; \mathrm{CH}_0(A), \qquad \forall j \in \mathbb{N},$$

where

$$\gamma(x) = \sum_{k=1}^{n} \frac{1}{k}(\{x\} - \{0_A\})^{*k}$$

is the logarithm of $\{x\}$. Since $\exp(\gamma(x)) = \{x\}$ and we can assume that $\Theta^g/g! = d\{0_A\}$ for some positive integer d, we get

$$\sum_{i=1}^{m} \{x_i\} = \sum_{i=1}^{m} \exp(\gamma(x_i)) = \sum_{i=1}^{m} \exp(\gamma(y_i)) = \sum_{i=1}^{m} \{y_i\} \in \mathrm{CH}_0(A).$$

□

Instead of considering the filtration $F^\bullet_{BB}$, we can consider the Beauville filtration $F^\bullet_B$ on the Chow ring of an abelian variety (see [Bea86]) to obtain an unconditional proof of an analogue of Conjecture 1.2:

Theorem 2.3. *Let A be an abelian g-fold and let $x_1, \cdots, x_m, y_1, \cdots, y_m \in A$,*

$$\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i \in \bigoplus_{s=m+1}^{g} \mathrm{CH}^g_{(s)}(A) \iff \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i \in \mathrm{CH}_0(A).$$

Proof. The map $x \mapsto D^j_x \in \mathrm{CH}^j(A)$ as above is a given by a correspondence $\Gamma_j \in \mathrm{CH}^j(A \times A)$. Since $\mathrm{CH}^i_{(s)}(A) = 0$ for all $s > i$,

$$\Gamma_{j*}(\mathrm{CH}^g_{(s)}(A)) = 0, \qquad \forall s \geq j.$$

Thus, if $\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i \in \bigoplus_{s=m+1}^{g} \mathrm{CH}^g_{(s)}(A)$,

$$\sum_{i=1}^{m} D^j_{x_i} = \sum_{i=1}^{m} D^j_{y_i}, \qquad \forall j \in \{1, \cdots, m\}.$$

We can then apply the same reasoning as in the proof of Proposition 2.2. □

Remark 2.4. The condition

$$\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i \in \bigoplus_{s=m+1}^{g} \mathrm{CH}^g_{(s)}(A)$$

appearing in Theorem 2.3 is a priori stronger than the condition

$$\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i \in F^{m+1}_B \mathrm{CH}_0(X).$$

They are equivalent assuming a conjecture of Beauville [Bea86] which states that $\mathrm{CH}^\bullet_{(s)}(A) = 0$ for all $s < 0$.

2.2. **Deducing Conjecture 1.2 for hyper-Kähler varieties from well-known conjectures on algebraic cycles.** In [Voi22], the author shows how the nilpotence conjecture implies Conjecture 1.1 for hyper-Kähler varieties satisfying the Lefschetz standard conjecture in degree 2. In Remark 2.14 she specifies that her proof does not imply a stronger version of the statement of Conjecture 1.2, where the depth in the Bloch-Beilinson does not depend on m, the degree of the effective zero-cycles. In short the reason is that $(x-y)\in\mathbb{Z}[x,y]$ divides (x^n-y^n) for all $n>0$, whereas

$$\sum_{i=1}^m x_i-\sum_{i=1}^m y_i \ \in\ \mathbb{Q}[x_1,\cdots,x_m,y_1,\cdots,y_m]$$

need not divide

$$\sum_{i=1}^m x_i^n-\sum_{i=1}^m y_i^n.$$

Nonetheless, one can adapt Voisin's argument using the fundamental theorem on symmetric polynomials and Conjecture 2.1. The main consequence of Conjecture 2.1 we will need is the following:

Lemma 2.5 ([Voi22] Lemma 2.8)**.** *If Conjecture 2.1 holds, then for any smooth projective varieties X,Y, correspondence $\Gamma\in\mathrm{CH}^i(X\times Y)$, and $w\in F_{BB}^{i+1}\mathrm{CH}_0(X)$, we have $\Gamma_*w=0$.*

Proposition 2.6. *Let X be a hyper-Kähler variety which satisfies Conjecture 2.1 and the Lefschetz standard conjecture in degree 2. Then the nilpotence conjecture implies Conjecture 1.2 for X.*

Proof. This proof is a simple adaptation of Voisin's argument in Section 2.2 of [Voi22] and we use the same notation. X is a smooth projective hyper-Kähler variety of dimension $2n$ and we fix a polarizing class $h_X\in\mathrm{Pic}(X)$. We let $Z_{\mathrm{lef}}\in\mathrm{CH}^2(X\times X)$ be a cycle whose existence is predicted by the Lefschetz standard conjecture in degree 2, namely such that $[Z_{\mathrm{lef}}]^*:H^{4n-2}(X)\longrightarrow H^2(X)$ is the inverse of the cup product map with h_X^{2n-2}. As shown in [Voi22], there is a polynomial in Z_{lef} and $\mathrm{pr}_2^*h_X$

$$P\ =\ \sum_{i=0}^n \mu_i Z_{\mathrm{lef}}^i\cdot\mathrm{pr}_2^*h_X^{2n-2i}\ \in\ \mathrm{CH}^{2n}(X\times X)$$

such that $[P]^*$ acts as the identity on holomorphic forms. Reasoning as in [Voi22] and assuming the nilpotence conjecture, one sees that the map $P_*:\mathrm{CH}_0(X)\longrightarrow\mathrm{CH}_0(X)$ is the identity.

Suppose that $x_1,\cdots,x_m,y_1,\cdots,y_m\in X$ and

$$\sum_{i=1}^m x_i-\sum_{i=1}^m y_i\in F^{2m+1}\mathrm{CH}_0(X).$$

Then by Conjecture 2.1

$$\sum_{i=1}^m (Z_{\mathrm{lef}}^j)_{x_i}\ =\ \sum_{i=1}^m (Z_{\mathrm{lef}}^j)_{y_i}\ \in\ \mathrm{CH}^{2i}(X),\qquad j=1,\cdots,m.$$

Denoting by $\iota_x:X\longrightarrow X\times X$ the embedding given by $\iota_x(y)=(x,y)$, we have

$$\begin{aligned}Z_{\mathrm{lef}}^j\cdot\{x\}\times X&=Z_{\mathrm{lef}}^j\cdot\iota_{x*}(X)=\iota_{x*}(\iota_x^*(Z_{\mathrm{lef}}^j))=\iota_{x*}(\iota_x^*(Z_{\mathrm{lef}})^j)\\&=\iota_{x*}\big((Z_{\mathrm{lef},x})^j\big)=\mathrm{pr}_1^*(\{x\})\cdot\mathrm{pr}_2^*\big((Z_{\mathrm{lef},x})^j\big).\end{aligned}$$

Accordingly,

$$(Z_{\mathrm{lef}}^j)_x=(Z_{\mathrm{lef},x})^j,\qquad j=0,\cdots,n.$$

This gives

$$\sum_{i=1}^m (Z_{\mathrm{lef},x_i})^j=\sum_{i=1}^m (Z_{\mathrm{lef},y_i})^j,\qquad j=0,\cdots,m.$$

By the fundamental theorem on symmetric polynomials the same equality holds for all j and thus

$$\sum_{i=1}^{m}(Z_{\mathrm{lef}}^{j})_{x_i}=\sum_{i=1}^{m}(Z_{\mathrm{lef}}^{j})_{y_i},\qquad j=0,\cdots,n.$$

Finally, we conclude that

$$\sum_{i=1}^{m}x_i=P_*\left(\sum_{i=1}^{m}x_i\right)=P_*\left(\sum_{i=1}^{m}y_i\right)=\sum_{i=1}^{m}y_i\ \in\ \mathrm{CH}_0(X).$$

□

3. Polynomial decompositions of the diagonal

3.1. **Definition and examples.** In [Voi22], Voisin formulates the following conjecture, which, together with Conjecture 2.1, implies Conjecture 1.2 for hyper-Kähler varieties:

Conjecture 3.1 (Conjecture 2.16 of [Voi22])**.** *Consider a smooth projective hyper-Kähler $2n$-fold X, a polarization h_X, and a cycle $Z_{\mathrm{lef}}\in\mathrm{CH}^2(X\times X)$ such that $[Z_{\mathrm{lef}}]^*$ is the inverse of the cup product with h_X^{2n-2}. There are cycles $\gamma_i\in\mathrm{CH}^{2n-2i}(X)$, $i=0,\cdots,n$, a divisor $D\subset X$, and a cycle $W\in\mathrm{CH}^{2n}(X\times X)$ supported on $D\times X$ such that*

$$\Delta_X=\sum_{i=0}^{n}Z_{\mathrm{lef}}^{i}\cdot\mathrm{pr}_2^*(\gamma_i)+W\in\mathrm{CH}^{2n}(X\times X).$$

This conjecture suggests the following definition:

Definition 3.2. *A smooth projective variety X admits a degree l polynomial decomposition of the diagonal up to coniveau c if*

$$\Delta_X=Z_1+Z_2\in\mathrm{CH}^n(X\times X),$$

where Z_1 belongs to the subalgebra of $\mathrm{CH}^\bullet(X\times X)$ generated in degree at most l and Z_2 is supported on $Y\times X$ for some closed $Y\subset X$ of codimension c. If c can be taken greater than $\dim X$ we say that X has a degree l polynomial decomposition of the diagonal.

One can define analogously the notion of integral polynomial decomposition of the diagonal by working with Chow rings with integral coefficients. We will only be concerned with rational polynomial decompositions.

Example 3.3.

(1) $\mathbb{P}^n$ *has a degree 1 polynomial decomposition of the diagonal. Indeed, writing h for $c_1(\mathcal{O}(1))$ we have*

$$\Delta_{\mathbb{P}^n}=\sum_{i=0}^{n}\mathrm{pr}_1^*(h^i)\cdot\mathrm{pr}_2^*(h^{n-i})\in\mathrm{CH}^n(\mathbb{P}^n\times\mathbb{P}^n).$$

(2) *A variety with a rational decomposition of the diagonal in the sense of Bloch-Srinivas* [BS83] *has a polynomial decomposition of the diagonal in degree 1 up to coniveau 1.*

(3) *Curves have a degree 1 polynomial decomposition of the diagonal. More generally, any n-fold has a degree n polynomial decomposition of the diagonal.*

(4) *A surface X with $p_g(X)=0$ has a degree 1 polynomial decomposition of the diagonal if and only if it satisfies Bloch's conjecture. Murre* [Mur90] *defines a decomposition of the motive $h(X)$*

$$h(X)=\sum_{i=0}^{4}h^i(X).$$

The motives $h^i(X)$, $i\neq 2$ are cut out by idempotent correspondences which are products of divisors. The motive $h^2(X)$ further breaks up as a sum $h^2_{\mathrm{alg}}(X)+h^2_{\mathrm{tr}}(X)$ [KMP07]*. If*

Bloch's conjecture holds for X, then $h^2_{\mathrm{tr}}(X) = 0$ and thus Δ is a polynomial in divisor classes. We will explain the converse in Proposition 3.9.

(5) *An abelian variety A has a degree 1 polynomial decomposition of the diagonal. Indeed, the diagonal of A is a rational multiple of $f^*(\Theta)^{\dim A}$, where $f : A^2 \longrightarrow A$ is given by $f(a,b) = a - b$, and Θ is a symmetric ample divisor.*

(6) *If $X_1, \cdots, X_r$ have polynomial decompositions of degree $l_1, \cdots l_r$ up to coniveau $c_1, \cdots, c_r$ then $X_1 \times \cdots \times X_r$ has a degree $\max_{1\leq i\leq r}(l_i)$ polynomial decomposition of the diagonal up to coniveau $\max_{1\leq i\leq r}(c_i)$.*

Example 3.4. *Let X be a smooth projective variety with $H^0(X, \Omega^1) = 0$. Since*

$$\mathrm{Pic}(X \times X) = \mathrm{pr}_1^*\mathrm{Pic}(X) \oplus \mathrm{pr}_2^*\mathrm{Pic}(X),$$

X has a degree 1 polynomial decomposition decomposition of the diagonal up to coniveau 1 if and only if X has a rational decomposition of the diagonal. By Proposition 1 of [BS83] *this is the case if and only if the degree map $\mathrm{CH}_0(X) \longrightarrow \mathbb{Q}$ is an isomorphism.*

The generalized Bloch conjecture predicts when the degree map $\mathrm{CH}_0(X) \longrightarrow \mathbb{Q}$ is an isomorphism.

Conjecture 3.5 (generalized Bloch conjecture [Voi02])**.** *Let X be a smooth projective n-fold such that $H^{p,q}(X) = 0$ for all $p \neq q$ and $p < c$. Then the cycle class map*

$$cl : \mathrm{CH}_i(X) \longrightarrow H^{2n-2i}(X, \mathbb{Q})$$

is injective for all $i < c$.

3.2. **Conjecture and relationship with the generalized Bloch conjecture.** Example 3.4 explains how the generalized Bloch conjecture predicts that a variety with $H^0(X, \Omega^1) = 0$ has a degree 1 polynomial decomposition of the diagonal up to coniveau 1 if and only if

$$H^\bullet(X, \mathbb{Q}) = N^1 H^\bullet(X, \mathbb{Q}),$$

where $N^\bullet$ denotes the Hodge coniveau filtration. Recall that $N^r H^\bullet(X, \mathbb{Q})$ is by definition the largest Hodge substructure $V \subset H^\bullet(X, \mathbb{Q})$ which has coniveau at least r, namely such that $V^{p,q}_{\mathbb{C}} = 0$ if $p < r$.

This provides evidence that a polynomial decomposition of the diagonal can be detected by Hodge theory.

Conjecture 3.6. *Let X be a smooth projective n-fold. Then X admits a degree l polynomial decomposition of the diagonal up to coniveau c if and only if*

$$N^r H^\bullet(X, \mathbb{Q})/N^{r+1} H^\bullet(X, \mathbb{Q})$$

is generated in degree at most $l + r$ for all $r < c$.

Remark 3.7. It suffices to check this in degree $\leq n$ as the hard Lefschetz theorem then implies the same generation statement in high degree. Example 3.3 (4) illustrates why the shift in degree of generation according to coniveau is necessary. For example, a smooth cubic surface has a degree 1 polynomial decomposition of the diagonal but its primitive cohomology is not generated in degree 1.

As is often the case, it is easy to deduce Hodge-theoretic information from cycle-theoretic information:

Proposition 3.8. *If X has a degree l decomposition of the diagonal up to coniveau c the algebra*

$$N^r H^\bullet(X, \mathbb{Q})/N^{r+1} H^\bullet(X, \mathbb{Q})$$

is generated in degrees at most $l + r$ for all $r < c$.

Proof. Observe that the subspace of $H^d(X,\mathbb{Q})$ generated in degrees at most $l+r$ is a sub-Hodge structure whose complexification is the subspace of $H^d(X,\mathbb{C})$ generated in degrees at most $l+r$. Consider a simple Hodge structure

$$V \subset N^r H^d(X,\mathbb{Q})$$

which is not contained in $N^{r+1}H^d(X,\mathbb{Q})$ and cycles $W_1,\cdots,W_k \subset X\times X$ of codimension $d_1,\cdots,d_k \leq l$ with

$$n =_{\text{def}} \dim X = d_1 + \cdots + d_k.$$

For each i, decompose the cycle class of W_i into Künneth components

$$[W_i] = \sum_{a_i,b_i=0}^{d_i} v_i^{a_i,b_i} \otimes w_i^{d_i-a_i,d_i-b_i},$$

where

$$v_i^{a_i,b_i} \otimes w_i^{d_i-a_i,d_i-b_i} \in H^{a_i,b_i}(X) \otimes H^{d_i-a_i,d_i-b_i}(X).$$

Given a non-zero class $\alpha \in V_{\mathbb{C}}^{r,d-r}$, we have

$$[W_1\cdots W_k]^*\alpha = \sum_{|\mathbf{a}|=r,|\mathbf{b}|=d-r} \left[\prod_{i=1}^{k}(v_i^{a_i,b_i} \otimes w_i^{d_i-a_i,d_i-b_i})\right]^* \alpha$$

where the sum is over tuples $\mathbf{a}=(a_1,\cdots,a_k)$, $\mathbf{b}=(b_1,\cdots b_k)$ with sums r and $d-r$ respectively. Since

$$(v_1^{a_1,b_1} \otimes w_1^{d_1-a_1,d_1-b_1} \cdots v_k^{a_k,b_k} \otimes w_k^{d_k-a_k,d_k-b_k})^*\alpha$$

is a multiple of $v_1^{a_1,b_1}\cdots v_k^{a_k,b_k}$ and $a_i \leq r$, $b_i \leq d_i \leq l$, we must have $a_i+b_i \leq l+r$. This shows that $[W_1\cdots W_k]^*\alpha$ is in the subspace of $H^d(X,\mathbb{C})$ generated in degree $\leq l+r$, and thus that V is contained in the subspace of $H^d(X,\mathbb{Q})$ generated in degree $\leq l+r$. □

Proposition 3.9. *Let X is a smooth projective n-fold with $H^0(X,\Omega^p)=0$ for all $p\geq 2$. If X has a degree 1 polynomial decomposition of the diagonal up to coniveau 1 then $F^2\mathrm{CH}_0(X)=0$. In particular, Conjecture 3.6 for $c=1$ implies the generalized Bloch conjecture for $c=1$.*

Proof. Since X has a degree 1 polynomial decomposition of the diagonal up to coniveau 1 we can write

$$\Delta_X = Z_1 + Z_2 \in \mathrm{CH}^n(X\times X),$$

where Z_1 is a polynomial in divisors and Z_2 is supported on $Y\times X$ for some divisors $Y\subset X$. Accordingly,

$$\mathrm{id}_{\mathrm{CH}_0(X)} = \Delta_{X*} = Z_{1*} : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(X).$$

We will show that any monomial $D_1\cdots D_n$, where $D_i \in \mathrm{Pic}(X\times X)$, gives a zero map on $F^2\mathrm{CH}_0(X)$ which will imply that $F^2\mathrm{CH}_0(X)=0$. Observe that

$$\mathrm{Pic}(X\times X) = \mathrm{pr}_1^*\mathrm{Pic}(X) \oplus \mathrm{pr}_2^*\mathrm{Pic}(X) \oplus H,$$

where H is the group that consists of pullbacks of divisors classes on $\mathrm{Alb}(X)\times\mathrm{Alb}(X)$ which have trivial restriction to the factors.

Given a monomial $D_1\cdots D_n$, if one of the D_i is in $\mathrm{pr}_1^*\mathrm{Pic}(X)$ then the map

$$(D_1\cdots D_n)_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(X)$$

is identically zero. Similarly, if at least $n-1$ of the D_i belong to $\mathrm{pr}_2^*\mathrm{Pic}(X)$ then the same map factors through the Chow group of zero-cycles on a variety of dimension 1 so that the induced map

$$(D_1\cdots D_n)_* : F^2\mathrm{CH}_0(X) \longrightarrow F^2\mathrm{CH}_0(X)$$

is identically zero.

Finally, suppose that $D_1, \cdots, D_i$ belong to $\mathrm{pr}_2^*\mathrm{Pic}(X)$ and that $D_{i+1}, \cdots, D_n$ are pulled back from divisors $D'_{i+1}, \cdots, D'_n$ on $\mathrm{Alb}(X) \times \mathrm{Alb}(X)$ under $\alpha \times \alpha$, where $\alpha : X \longrightarrow \mathrm{Alb}(X)$ is the Albanese morphism. Since $H^0(X, \Omega^2) = 0$, the Albanese dimension of X is at most 1. We can assume that the image of X in its Albanese is a curve C, or else $H = 0$ and we are done by the considerations above. Since $D_{i+1} \cdot D_n = 0$ if $i < n-2$, it suffices to consider the case $i = n-2$. Then $D_{n-1} \cdot D_n$ is the pullback of a zero cycle on $C \times C$. Since the pullback of a point on $C \times C$ is the product of a divisor in $\mathrm{pr}_1^*\mathrm{Pic}(X)$ and a divisor in $\mathrm{pr}_2^*\mathrm{Pic}(X)$, we see that $D_1 \cdots D_n$ induces the zero map on $\mathrm{CH}_0(X)$.

□

3.3. **Relation between Conjecture 3.6 and Conjecture 1.2.**

Proposition 3.10. *Conjecture 3.6 for $c = 1$ along with Conjecture 2.1 implies Conjecture 1.2.*

Proof. Suppose that X is such that $H^{\bullet,0}(X, \mathbb{C})$ is generated in degree at most l. Then

$$H^{\leq n}(X, \mathbb{Q})/N^1 H^{\leq n}(X, \mathbb{Q})$$

is generated in degrees $\leq l$. By Conjecture 3.6 $\Delta_X = Z_1 + Z_2$, where Z_1 is in the subalgebra of $\mathrm{CH}^\bullet(X \times X)$ generated in degrees $\leq l$, and Z_2 is supported on $Y \subset X$ for some proper closed subset $Y \subset X$. Hence,

$$\Delta_{X*} = Z_{1*} : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(X).$$

Consider a basis $W_1, \cdots, W_n$ for

$$\bigoplus_{i \leq l} \mathrm{CH}^i(X \times X).$$

In order to proceed, we will need a generalization of the fundamental theorem on symmetric polynomials. Consider variables $w_j^{(i)}$ where $1 \leq j \leq n$ and $1 \leq i \leq m$. The symmetric group $\mathfrak{S}_m$ acts on the ring $\mathbb{Q}[w_j^{(i)}]$ by permuting the superscript and the subalgebra $\mathbb{Q}[w_j^{(i)}]^{\mathfrak{S}_m}$ is called the subalgebra of multi-symmetric polynomials. Given an element $\mathbf{a} =_{\mathrm{def}} (a_1, \cdots, a_n) \in \mathbb{N}^n \setminus \mathbf{0}$, the corresponding power sum multisymmetric polynomial is

$$p_{\mathbf{a}}(w_j^{(i)}) = \sum_{s=1}^{m} {w_1^{(s)}}^{a_1} \cdots {w_n^{(s)}}^{a_n} \in \mathbb{Q}[w_j^{(i)}]^{\mathfrak{S}_m}.$$

It is a classical fact that $\mathbb{Q}[w_j^{(i)}]^{\mathfrak{S}_m}$ is generated by elementary multisymmetric power sums (see the references in the introduction of [Bri04]).

Proposition 3.11 ([Bri04] Corollary 5)**.** *All power sums are in the ideal generated by the powers sums $p_{\mathbf{a}}(w_j^{(i)})$ with $a_1 + \cdots + a_n \leq m$.*

Consider $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{N}^n \setminus \{\mathbf{0}\}$ with $a_1 + \cdots + a_n \leq m$. The cycle

$$P_{\mathbf{a}} =_{\mathrm{def}} W_1^{a_1} \cdots W_n^{a_n}$$

has codimension at most ml. Given $x_1, \cdots, x_m, y_1, \cdots, y_m \in X$ such that

$$\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i \in F_{BB}^{ml+1}\mathrm{CH}_0(X).$$

Conjecture 2.1 and Lemma 2.5 imply that

$$\sum_{i=1}^{m} P_{\mathbf{a},x_i} = \sum_{i=1}^{m} P_{\mathbf{a},y_i}.$$

The same argument as in the proof of Proposition 2.6 shows that

$$P_{\mathbf{a},x_i} = W_{1,x_i}^{a_1} \cdots W_{n,x_i}^{a_n} \in \mathrm{CH}^\bullet(X),$$

so that

$$\sum_{i=1}^{m} W_{1,x_i}^{a_1} \cdots W_{n,x_i}^{a_n} = \sum_{i=1}^{m} W_{1,y_i}^{a_1} \cdots W_{n,y_i}^{a_n} \in \mathrm{CH}^\bullet(X).$$

Since this holds for all $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{N}^n \setminus \{\mathbf{0}\}$ with $a_1 + \cdots + a_n \leq m$, applying Lemma 3.11 with $w_j^{(i)} = W_{j,x_i}$ gives

$$\sum_{i=1}^{m} P_{\mathbf{a},x_i} = \sum_{i=1}^{m} P_{\mathbf{a},y_i} \in \mathrm{CH}^\bullet(X)$$

for any $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{N}^n \setminus \{\mathbf{0}\}$. Since Δ is a polynomial in the W_i, it follows that

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \Delta_{x_i} = \sum_{i=1}^{m} \Delta_{y_i} = \sum_{i=1}^{m} y_i \in \mathrm{CH}_0(X).$$

□

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