

ON A CONJECTURE OF VOISIN ON THE GONALITY OF VERY GENERAL ABELIAN VARIETIES

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ABSTRACT. We adapt a method of Voisin to powers of abelian varieties in order to study orbits for rational equivalence of zero-cycles on very general abelian varieties. We deduce that a very general abelian variety of dimension at least $2k - 2$ has gonality at least $k + 1$. This settles a conjecture of Voisin. We also discuss how upper bounds for the dimension of orbits for rational equivalence can be used to provide new lower bounds on other measures of irrationality. In particular, we obtain a strengthening of the Sommese bound on the degree of irrationality of abelian varieties. In the appendix we present some new identities in the Chow group of zero-cycles of abelian varieties.

1. INTRODUCTION

In his seminal 1969 paper [8] Mumford shows that the Chow group of zero-cycles of a smooth projective surface over \mathbb{C} with $p_g > 0$ is very large. Building on the work of Mumford, in [10] and [11] Roitman studies the map

$$\mathrm{Sym}^k(X) \rightarrow CH_0(X)$$

for X a smooth complex projective variety. He shows that fibers of this map, which we call orbits of degree k for rational equivalence¹, are countable unions of Zariski closed subsets. Moreover, he defines birational invariants $d(X)$ and $j(X) \in \mathbb{Z}_{\geq 0}$ such that for $k \gg 0$ the minimal dimension of orbits of degree k for rational equivalence is $k(\dim X - d(X)) - j(X)$. Roitman's generalization of Mumford's theorem is the following statement:

$$\text{If } H^0(X, \Omega^q) \neq 0 \text{ then } d(X) \geq q.$$

In particular, if X has a global holomorphic top form then a very general $x_1 + \dots + x_k \in \mathrm{Sym}^k X$ is contained in a zero-dimensional orbit.

Abelian varieties are among the simplest examples of varieties admitting a global holomorphic top form. In this article we will focus our attention on this example and take a point of view different from the one mentioned above. Instead of fixing an abelian variety A and considering the minimal dimension of orbits of degree k for rational equivalence, we will be interested in the maximal dimension of such orbits for a very general abelian variety A of dimension g with a given polarization θ .

This perspective has already been studied by Pirola, Alzati-Pirola, and Voisin (see respectively [9],[1],[14]) with a view towards the gonality of very general abelian varieties.

¹For simplicity we will call both fibers of $X^k \rightarrow CH_0(X)$ and $\mathrm{Sym}^k X \rightarrow CH_0(X)$ orbits of degree k for rational equivalence.

The story begins with [9] in which Pirola shows that, given a very general abelian variety A , curves of geometric genus less than $\dim A - 1$ are rigid in the Kummer variety $K(A) = A/\{\pm 1\}$. This allows him to show:

Theorem 1.1 (Pirola). *A very general abelian variety of dimension ≥ 3 does not have positive dimensional orbits of degree 2. In particular it does not admit a non-constant morphism from a hyperelliptic curve.*

There are several ways in which one might hope to generalize this result. For instance, one can ask for the gonality of a very general abelian variety of dimension g . We define the gonality of a smooth projective variety X as the minimal gonality of the normalization of an irreducible curve $C \subset X$. Note that an irreducible curve $C \subset X$ whose normalization \tilde{C} has gonality k provides us with a positive dimensional orbit in $\text{Sym}^k \tilde{C}$, and thus with a positive dimensional orbit in $\text{Sym}^k X$. Hence, one can give a lower bound on the gonality of X by giving a lower bound on the degree of positive dimensional orbits. This suggest to consider the function

$$\mathcal{G}(k) := \min \left\{ g \in \mathbb{Z}_{>0} : \begin{array}{l} \text{a very general abelian variety of dimension } g \\ \text{does not have a positive dimensional orbit of degree } k \end{array} \right\},$$

and attempt to find an upper bound² on $\mathcal{G}(k)$. Indeed, a very general abelian variety of dimension at least $\mathcal{G}(k)$ must have gonality at least $k + 1$.

In this direction, a few years after the publication of [9], Alzati and Pirola improved on Pirola's results in [1], showing that a very general abelian variety A of dimension ≥ 4 does not have positive dimensional orbits of degree 3, i.e. that $\mathcal{G}(3) \leq 4$. This suggest that for any $k \in \mathbb{N}$ a very general abelian variety of sufficiently large dimension should not admit a positive dimensional orbit of degree k , i.e. that $\mathcal{G}(k)$ is finite for any $k \in \mathbb{Z}_{>0}$. This problem was posed in [2] and answered positively by Voisin in [14].

Theorem 1.2 (Voisin, Thm. 0.4 (ii) in [14]). *A very general abelian variety of dimension at least $2^k(2k - 1) + (2^k - 1)(k - 2)$ does not have a positive dimensional orbit of degree k , i.e. $\mathcal{G}(k) \leq 2^k(2k - 1) + (2^k - 1)(k - 2)$.*

Voisin provides some evidence suggesting that this bound can be improved significantly. Her main conjecture from [14] is the following linear bound on the gonality of very general abelian varieties:

Conjecture 1.3 (Voisin, Conj. 0.2 in [14]). *A very general abelian variety of dimension at least $2k - 1$ has gonality at least $k + 1$.*

The central result of this paper is the proof of this conjecture. It is obtained by generalizing Voisin's method to powers of abelian varieties. This allows us to rule out the existence of positive dimensional CCS_k in very general abelian varieties of dimension g for g large compared to k .

Theorem 4.4. *For $k \geq 3$, a very general abelian variety of dimension at least $2k - 2$ has no positive dimensional orbits of degree k , i.e. $\mathcal{G}(k) \leq 2k - 2$.*

²Note that

$$\mathbb{Z}_{\geq \mathcal{G}(k)} = \left\{ g \in \mathbb{Z}_{>0} : \begin{array}{l} \text{a very general abelian variety of dimension } g \\ \text{does not have a positive dimensional orbit of degree } k \end{array} \right\}.$$

To see this, observe that if a very general abelian variety A of dimension g has a positive dimensional orbit $Z \subset A^k$ of degree k , we can degenerate A to an abelian variety isogenous to a product $B \times E$ in such a way that the restriction of the projection $p : (B \times E)^k \rightarrow B^k$ to Z has a positive dimensional image.

This gives the following lower bound on the gonality of very general abelian varieties:

Corollary 4.5. *For $k \geq 3$, a very general abelian variety of dimension $\geq 2k - 2$ has gonality at least $k + 1$. In particular Conjecture 1.3 holds.*

Another approach to generalizing Theorem 1.1 is the following: Observe that a nonconstant morphism from a hyperelliptic curve C to an abelian surface A gives rise to a positive dimensional orbit of the form $2\{0_A\}$. Indeed, translating C if needed, we can assume that a Weierstrass point of C maps to 0_A . This suggests to consider the maximal g for which a very general abelian variety A of dimension g contains an irreducible curve $C \subset A$ whose normalization $\pi : \tilde{C} \rightarrow C$ admits a degree k morphism to \mathbb{P}^1 with a point of ramification index at least l . Here we say that $p \in \tilde{C}$ has ramification index at least l if the sum of the ramification indices of $\tilde{C} \rightarrow \mathbb{P}^1$ at all points in $\pi^{-1}(\pi(p))$ is at least l . This maximal g is less than

$$\mathcal{G}_l(k) := \min \left\{ \begin{array}{l} g \in \mathbb{Z}_{>0} : \text{a very general abelian variety of dimension } g \text{ does not} \\ \text{admit a positive dimensional orbit of the form } |\sum_{i=1}^{k-l} \{a_i\} + l\{a_0\}| \end{array} \right\},$$

where, given a smooth projective variety X and $\{x_1\} + \dots + \{x_k\} \in \text{Sym}^k X$, we denote by $|\{x_1\} + \dots + \{x_k\}|$ the orbit of $\{x_1\} + \dots + \{x_k\}$, namely the orbit containing this point. Clearly, we have

$$\mathcal{G}_1(k) = \mathcal{G}_0(k) = \mathcal{G}(k),$$

and

$$\mathcal{G}_l(k) \leq \mathcal{G}(k).$$

In [14] the author shows the following:

Theorem 1.4 (Voisin Thm. 0.4 (iii) & (iv)). *The following inequality holds:*

$$\mathcal{G}_l(k) \leq 2^{k-l}(2k - 1) + (2^{k-l} - 1)(k - 2).$$

In particular

$$\mathcal{G}_k(k) \leq 2k - 1.$$

We can improve on Voisin's result to show the following:

Theorem 4.7. *A very general abelian variety A of dimension at least $2k + 2 - l$ does not have a positive dimensional orbit of the form $|\sum_{i=1}^{k-l} \{a_i\} + l\{0_A\}|$, i.e. $\mathcal{G}_l(k) \leq 2k + 2 - l$. Moreover, if A is a very general abelian variety of dimension at least $k + 1$ the orbit $|k\{0_A\}|$ is countable, i.e. $\mathcal{G}_k(k) \leq k + 1$.*

Taking a slightly different perspective, one can consider the maximal dimension of orbits of degree k of abelian varieties. One of the main contributions of [14] is the following extension of results of Alzati-Pirola (the cases $k = 2, 3$):

Theorem 1.5. *Orbits of degree k on an abelian variety have dimension at most $k - 1$.*

As observed by Voisin, one sees easily³ that this bound cannot be improved if one considers all abelian varieties. Moreover, it cannot be improved for very general abelian surfaces as shown in Example 2.13. Nevertheless Theorem 1.2 shows that it can be improved for very general abelian varieties of large dimension. In this direction we have:

³Consider abelian varieties of dimension g isogenous to $B \times E$, where B is a $(g - 1)$ -dimensional abelian variety and E is an elliptic curve.

Theorem 5.2. *Orbits of degree k on a very general abelian variety of dimension $\geq k - 1$ have dimension at most $k - 2$.*

Recall that the degree of irrationality $\text{Irr}(X)$ of a variety X is the minimal degree of a dominant morphism $X \rightarrow \mathbb{P}^{\dim X}$. The previous theorem provides the following improvement of the Sommese bound $\text{Irr}(A) \geq \dim A + 1$ (see the appendix to [3]) on the degree of irrationality of abelian varieties.

Corollary 5.3. *If A is a very general abelian variety of dimension $g \geq 3$, then*

$$\text{Irr}(A) \geq g + 2.$$

It is very likely that one can do better by studying the Gauss map of CCS_k^4 of very general abelian varieties.

Finally, in the appendix we present the following generalization of Proposition 0.9 of [14] along with similar results:

Proposition A.2. *Consider an abelian variety A and effective zero-cycles $\sum_{i=1}^k \{x_i\}$, $\sum_{i=1}^k \{y_i\}$ on A such that*

$$\sum_{i=1}^k \{x_i\} = \sum_{i=1}^k \{y_i\} \in CH_0(A).$$

Then for $i = 1, \dots, k$

$$\prod_{j=1}^k (\{x_i\} - \{y_j\}) = 0 \in CH_0(A),$$

where the product is the Pontryagin product.

In the last part of this introduction we will sketch Voisin's proof of Theorem 1.2 and give an idea of the methods we will use to show Theorem 4.4. Voisin considers what she calls *naturally defined subsets of abelian varieties*. Given a universal abelian variety \mathcal{A}/S of dimension g with a fixed polarization θ there are subvarieties $S_\lambda \subset S$ along which $\mathcal{A}_s \sim \mathcal{B}_s^\lambda \times \mathcal{E}_s^\lambda$, where $\mathcal{B}^\lambda/S_\lambda$ is a family of abelian varieties of dimension $(g - 1)$ and $\mathcal{E}^\lambda/S_\lambda$ is a family of elliptic curves. Let $S_\lambda(B) = \{s \in S_\lambda : B_s^\lambda = B\}$.

Voisin shows that, given a naturally defined subset $\Sigma_{\mathcal{A}} \subsetneq \mathcal{A}$, there is an S_λ such that the image of the restriction of $p_\lambda : (\mathcal{B}_s^\lambda \times \mathcal{E}_s^\lambda)^k \rightarrow (\mathcal{B}_s^\lambda)^k = B^k$ to $\Sigma_{\mathcal{A}_s}$ varies with $s \in S_\lambda(B)$, for a generic B in the family \mathcal{B}^λ . This shows that if Σ_B is d -dimensional for a very-general $(g - 1)$ -dimensional abelian variety B with $0 < d < g - 1$, then $\Sigma_{\mathcal{A}}$ is at most $(d - 1)$ -dimensional for a very general abelian variety A of dimension g .

She proceeds to show that the set

$$\bigcup_{i=1}^k \text{pr}_i(|k\{0_A\}|) = \left\{ a_1 \in A : \exists a_2, \dots, a_k \in A \text{ such that } \sum_{i=1}^k \{a_i\} = k\{0_A\} \in CH_0(A) \right\}$$

⁴See Definition 2.2.

is contained in a naturally defined subset $A_k \subset A$ and that $\dim A_k \leq k - 1$. In particular, $A_k \neq A$ if $\dim A \geq k$. It follows from the discussion above that

$$\dim \bigcup_{i=1}^k \text{pr}_i(|k\{0_A\}|) = 0$$

if $\dim A \geq 2k - 1$ and A is very general. An induction argument finishes the proof. Voisin's results are very similar in spirit to those of [1] but a key difference is that the latter is concerned with subvarieties of A^k and not of A . We will see that this has important technical consequences. While Lemma 1.5 in [14] shows that the restriction of the projection $B \times E \rightarrow B$ to $\Sigma_{B \times E}$ is generically finite on its image, this becomes a serious sticking point in [1] (see lemmas 6.8 to 6.10 of [1]). One of the innovations of this article is a way to bypass this generic finiteness assumptions by using the inductive nature of the argument.

Acknowledgements. This paper owes a lot to the work of Alzati, Pirola, and Voisin. I thank Madhav Nori and Alexander Beilinson for countless useful and insightful conversations as well as for their support. I would also like to extend my gratitude to Claire Voisin for bringing my attention to this circle of ideas by writing [14], and for kindly answering some questions about the proof of Theorem 0.1 from that article.

2. PRELIMINARIES

In this section we fix some notation and establish some facts about positive dimensional orbits for rational equivalence on abelian varieties.

A variety will mean a quasi-projective reduced scheme of finite type over \mathbb{C} . In what follows, X is a smooth projective variety, \mathcal{X}/S is a family of such varieties, A is an abelian variety of dimension g with polarization type θ , and \mathcal{A}/S is a family of such abelian varieties. Mostly we will be concerned with locally complete families of abelian varieties, namely families \mathcal{A}/S such that the corresponding morphism from S to the moduli stack of g -dimensional abelian varieties with polarization type θ is dominant. In an effort to simplify notation, we will often write \mathcal{X}^k instead of \mathcal{X}_S^k to denote the k -fold fiber product of \mathcal{X} with itself over S . $\mathcal{Z} \subset \mathcal{A}^k$ will be a subvariety such that $\mathcal{Z} \rightarrow S$ is flat with irreducible fibers \mathcal{Z}_s . Finally, $CH_0(X)$ will denote the Chow group of zero-cycles of a smooth projective variety X with \mathbb{Q} -coefficients.

Remark 2.1. In many of our arguments, we will have a family of varieties $\mathcal{X} \rightarrow S$ and a subvariety $\mathcal{Z} \subset \mathcal{X}$, such that $\mathcal{Z} \rightarrow S$ is flat with irreducible fibers. We will often need to base change by a generically finite morphism $S' \rightarrow S$. To avoid the growth of notation we will denote the base changed family by $\mathcal{X} \rightarrow S$ again. Moreover, if $S_\lambda \subset S$ is a Zariski closed subset, the base change of S_λ under $S' \rightarrow S$ will also be denoted S_λ . Note that this applies also to the statement of theorems. For instance if we say a statement holds for a family \mathcal{X}/S we mean that it holds for some $\mathcal{X}_{S'}/S'$, where $S' \rightarrow S$ is generically finite.

Instead of considering orbits for rational equivalence, one can consider subvarieties of orbits. This makes talking about families of orbits somewhat simpler.

Definition 2.2. *A constant cycle subvariety of degree k (CCS_k) of X is a Zariski closed subset of X^k contained in a fiber of $X^k \rightarrow CH_0(X)$. A Zariski closed subset $\mathcal{Z} \subset \mathcal{X}^k$ is a CCS_k/S if \mathcal{Z}_s is a CCS_k of \mathcal{X}_s^k for every $s \in S$.*

The notion of CCS_k above is closely related but not to be confused with constant cycle subvarieties in the sense of Huybrechts (see [7]). Indeed a CCS_1 is exactly the analogue of a constant cycle subvariety as defined in [7] for K3 surfaces. Nonetheless a CCS_k of X need not be a CCS_1 of X^k ; in the first case we consider rational equivalence of cycles in X , and in the other rational equivalence of cycles in X^k . We will not only be interested in CCS_k but in families of CCS_k and subvarieties of X^k foliated by CCS_k .

Definition 2.3.

- (1) An $(r + d)$ -dimensional Zariski closed subset $Z \subset X^k$ is foliated by d -dimensional CCS_k if for all $z \in Z$ we have

$$\dim |z| \cap Z \geq d.$$

- (2) Similarly $\mathcal{Z} \subset \mathcal{X}^k$ is foliated by d -dimensional CCS_k if \mathcal{Z}_s is foliated by d -dimensional CCS_k for all $s \in S$.
- (3) An r -parameter family of d -dimensional CCS_k of X is an r -dimensional locally closed subset D of a Chow variety of X^k parametrizing dimension d cycles with a fixed cycle class, such that for each $t \in D$ the corresponding cycle is a CCS_k .
- (4) Similarly, an r -parameter family of d -dimensional CCS_k of \mathcal{X}/S is an r -dimensional locally closed subset \mathcal{D} of the relative Chow variety of \mathcal{X}^k parametrizing dimension d cycles with a fixed cycle class, such that for each $t \in \mathcal{D}$ the fiber of the corresponding cycle over any $s \in S$ is a CCS_k .

Note that given D , an r -parameter family of d -dimensional CCS_k of a variety X , and $\mathcal{Y} \rightarrow D$, the corresponding family of cycles, the set $\bigcup_{t \in D} \mathcal{Y}_t \subset X^k$ is foliated by d -dimensional CCS_k . Yet its dimension might be less than $(r + d)$. Conversely, any subvariety of X^k foliated by positive dimensional CCS_k arises in such a fashion from a family of CCS_k after possibly passing to a Zariski open subset. Indeed, by work of Roitman (see [10]) the set

$$\Delta_{rat} = \left\{ ((x_1, \dots, x_k), (x'_1, \dots, x'_k)) \in X^k \times X^k : \sum_{i=1}^k x_i = \sum_{i=1}^k x'_i \in CH_0(X) \right\} \subset X^k \times X^k$$

is a countable union of Zariski closed subsets. Consider

$$\pi : \Delta_{rat} \cap Z \times Z \rightarrow Z.$$

Given an irreducible component Δ' of $\Delta_{rat} \cap Z \times Z$ that dominates Z and has relative dimension d over Z , we can consider the map from an open set in Z to an appropriate Chow variety of X^k taking $z \in Z$ to the cycle $[\Delta'_z]$. Letting D be the image of this morphism, we get a family of CCS_k with the desired property.

Given an abelian variety A , we denote by A_M^r (or A_M , when $r = 1$) the image of A^r in A^k under the embedding $i_M : A^r \rightarrow A^k$ given by

$$(a_1, \dots, a_r) \mapsto \left(\sum_{j=1}^r m_{1j} a_j, \sum_{j=1}^r m_{2j} a_j, \dots, \sum_{j=1}^r m_{kj} a_j \right),$$

where $0 \leq r \leq k$, and $M = (m_{ij}) \in M_{k \times r}(\mathbb{Z})$ has rank r . We will use the same notation V_M^r , with $M \in M_{k \times r}(\mathbb{C})$, for V a vector space over \mathbb{C} . If A is simple then all abelian subvarieties

of A^k are of this form. Let $\text{pr}_i : A^k \rightarrow A$ be the projections to the i^{th} factor and, given a form $\omega \in H^0(A, \Omega^q)$, let $\omega_k := \sum_{i=1}^k \text{pr}_i^* \omega$.

For the sake of simplicity we mostly deal with X^k rather than $\text{Sym}^k X$ and we take the liberty to call points of X^k effective zero-cycles of degree k .

Definition 2.4. *Given an abelian variety A , a zero-cycle $z = (z_1, \dots, z_k) \in A^k$ is called normalized if $z_1 + \dots + z_k = 0_A$. We write $A^{k,0}$ for the kernel of the summation map, i.e. the set of normalized effective zero-cycles of degree k .*

In Corollary 3.5 of [1] the authors show the following generalization of Mumford's result:

Proposition 2.5 (Alzati-Pirola, Corollary 3.5 in [1]). *Let D be an r -parameter family of CCS_k of a variety X with corresponding family of cycles $Z \rightarrow D$. Denote by $g : Z \rightarrow X^k$ the natural map. If $\omega \in H^0(A, \Omega^q)$ and $q > r$, then*

$$g^*(\omega_k) = 0.$$

In proposition 3.2 of [14], Voisin shows that if A is an abelian variety, and $Z \subset A^k$ is such that $\omega_k|_Z = 0$ for all $\omega \in H^0(A, \Omega^q)$ and all $q \geq 1$, then $\dim Z \leq k - 1$. Along with the above proposition this gives:

Corollary 2.6. *An abelian variety does not admit a one-parameter family of $(k - 1)$ -dimensional CCS_k .*

This non-existence result along with our degeneration and projection argument will provide the proof of Theorem 5.2. For most other applications we will use non-existence results for large families of CCS_k on surfaces X with $p_g \neq 0$. In particular we have:

Lemma 2.7. *Let X be a smooth projective surface with $p_g \neq 0$ and $\omega \neq 0 \in H^0(X, \Omega_X^2)$. If $Z \subset X^k$ is a Zariski closed subset of dimension m foliated by d -dimensional CCS_k then $\omega_k^{[(m-d+1)/2]}$ restricts to zero on Z .*

Proof. The set of points $z \in Z$ such that $z \in Z_{sm} \cap (|z| \cap Z)_{sm}$ is clearly Zariski dense. Thus it suffices to show that $\omega_k^{[(m-d+1)/2]}$ restricts to zero on $T_{Z,z}$ for such a z . Suppose that $m - d$ is odd (the even case is treated in the same way). Given any $(m - d + 1)$ -dimensional subspace of $T_{Z,z}$, it must meet the tangent space to $T_{|z| \cap Z, z}$ and so we can assume it admits a basis v_1, \dots, v_{m-d+1} with $v_1 \in T_{|z| \cap Z, z}$. Hence

$$\iota_{v_1, \dots, v_{m-d+1}} \omega_k^{(m-d+1)/2}$$

will consist of a product of terms of the form $\iota_{v_i, v_j} \omega_k$. But $\iota_{v_1, v_j} \omega_k = 0$ for any j by Proposition 2.5. Indeed for any j the space spanned by v_1 and v_j is contained in the tangent space to a $(d + 1)$ -fold foliated by at least d -dimensional CCS_k . \square

Lemma 2.8. *Let X and ω be as in the previous lemma. If $Z \subset X^k$ is such that ω_k^l restricts to zero on Z , then $\dim Z < k + l$.*

Proof. Pick $z \in Z_{sm}$ and let v_1, \dots, v_m be a basis of $T_{Z,z}$. Complete it with v_{m+1}, \dots, v_{2k} to a basis of $T_{X^k, z}$. Let ω be a symplectic form on X . Since ω_k^k is a volume form on X^k we have $\iota_{v_1, \dots, v_{2k}} \omega_k^k \neq 0$. If $m \geq k + l$ then $\iota_{v_{m+1}, \dots, v_{2k}} \omega_k^k \in (\omega_k^l) \subset H^0(X^k, \Omega_{X^k}^l)$. The non-vanishing stated above then implies ω_k^l cannot vanish on Z . \square

Lemma 2.9. *If A is an abelian variety and $\omega \neq 0 \in H^0(A, \Omega_A^2)$, then*

$$(i_M^* \omega_k)^\wedge \neq 0 \in H^0(A_M^l, \Omega_{A_M^l}^{2l}).$$

In particular, if $\dim A = 2$, the form ω_k restricts to a symplectic form on A_M^l .

Proof. Let z, w be two coordinates on A and $\omega = dz \wedge dw$. Let z_i, w_i be the corresponding coordinates on the i^{th} factor of A^l . We have

$$i_M^* \omega_k = \sum_{i=1}^k \left(\sum_{j=1}^l m_{ij} dz_j \wedge \sum_{j'=1}^l m_{ij'} dw_{j'} \right).$$

We claim that

$$(i_M^* \omega_k)^\wedge = l! \det \mathbf{G} dz_1 \wedge dw_1 \wedge \dots \wedge dz_l \wedge dw_l,$$

where $\mathbf{G} = (\langle M_i, M_j \rangle)_{1 \leq i, j \leq l}$ is the Gram matrix of the columns M_i of the matrix M . Since M has maximal rank its Gram matrix has positive determinant. It follows that $(i_M^* \omega_k)^\wedge \neq 0$. To prove the above claim we observe that

$$i_M^* \omega_k = \sum_{i=1}^k \sum_{j, j'=1}^l m_{ij} m_{ij'} dz_j \wedge dw_{j'} = \sum_{j, j'=1}^l \langle M_j, M_{j'} \rangle dz_j \wedge dw_{j'}.$$

It follows that

$$\begin{aligned} (i_M^* \omega_k)^\wedge &= \sum_{\sigma, \sigma' \in S_l} \prod_{j=1}^l \langle M_{\sigma(j)}, M_{\sigma'(j)} \rangle dz_{\sigma(1)} \wedge dw_{\sigma'(1)} \wedge \dots \wedge dz_{\sigma(l)} \wedge dw_{\sigma'(l)} \\ &= \sum_{\sigma, \sigma' \in S_l} \text{sgn}(\sigma) \text{sgn}(\sigma') \prod_{j=1}^l \langle M_{\sigma(j)}, M_{\sigma'(j)} \rangle dz_1 \wedge dw_1 \wedge \dots \wedge dz_l \wedge dw_l \\ &= \sum_{\sigma' \in S_l} \text{sgn}(\sigma') \sum_{\sigma \in S_l} \text{sgn}(\sigma' \sigma) \prod_{j=1}^l \langle M_{\sigma' \sigma(j)}, M_{\sigma'(j)} \rangle dz_1 \wedge dw_1 \wedge \dots \wedge dz_l \wedge dw_l \\ &= \sum_{\sigma' \in S_l} \sum_{\sigma' \sigma \in S_l} \text{sgn}(\sigma) \prod_{j=1}^l \langle M_{\sigma(j)}, M_j \rangle dz_1 \wedge dw_1 \wedge \dots \wedge dz_l \wedge dw_l \\ &= l! \det \mathbf{G} \end{aligned}$$

□

In particular Lemma 2.7 implies:

Corollary 2.10. *If $Z \subset A^k$ is foliated by positive dimensional CCS_k and $\dim A = 2$ then it cannot be an abelian subvariety of the form A_M^l .*

This corollary will play a crucial technical role in our argument. Using Lemma 2.8, we see that the proof of Lemma 2.5 also shows the following:

Lemma 2.11. *If A is an abelian surface and $Z \subset A^{k,0}$ is such that ω_k^l vanishes on Z , then*

$$\dim Z < k + l - 1.$$

Corollary 2.12. *If A is an abelian surface and $Z \subset A^{k,0}$ is foliated by d -dimensional CCS_k , then*

$$\dim Z \leq 2(k-1) - d.$$

Proof. By Lemma 2.7 $\omega^{\lceil(\dim Z - d + 1)/2\rceil}$ restricts to zero on Z . Then by Lemma 2.8

$$\dim Z < k + \lceil(\dim Z - d + 1)/2\rceil - 1.$$

This gives the stated bound for parity reasons. \square

One could instead seek existence results for subvarieties of A^k foliated by d -dimensional CCS_k . Alzati and Pirola show in examples 5.2 and 5.3 of [1] that any abelian surface has a 2-dimensional CCS_3 and a 2-parameter family of normalized CCS_1 . In particular, using the argument from Remark 5.1 we see that Corollary 2.12 is sharp for $d = 0, 1, 2$.

Example 2.13. In [6] Lin shows that Corollary 2.12 is sharp for every d .

The methods of [6] can be used to show the following:

Proposition 2.14. *If an abelian variety A of dimension g is the quotient of the Jacobian of a smooth genus g' curve C , then A^k contains a $(g(k+1 - g' - d) + d)$ -dimensional subvariety foliated by d -dimensional normalized CCS_k for all $k \geq g' + d - 1$.*

Proof. To simplify notation we identify C with its image in $J(C)$. We can assume that $0_A \in C$. Recall that the summation map $\text{Sym}^l C \rightarrow J(C)$ has fibers $\mathbb{P}^{l-g'}$ for all $l \geq g'$. Moreover, if (c_1, \dots, c_l) and (c'_1, \dots, c'_l) are such that $\sum c_i = \sum c'_i \in J(C)$, then the zero cycles $\sum\{c_i\}$ and $\sum\{c'_i\}$ are equal in $CH_0(C)$ and thus in $CH_0(A)$.

In light of Remark 5.1, it suffices to show that $A^{g'+d-1,0}$ contains a d -dimensional $\text{CCS}_{g'+d-1}$. Consider the map

$$\psi : C \times C^{g'+d-1} \rightarrow A^{g'+d-1}$$

given by

$$(c_0, (c_1, \dots, c_{g'+d-1})) \mapsto (c_1 - c_0, \dots, c_{g'+d-1} - c_0).$$

This morphism is generically finite on its image since the restriction of the summation map $A^2 \rightarrow A$ to $C^2 \subset A^2$ is. The intersection of the image of ψ with $A^{g'+d-1,0}$ is d -dimensional and we claim it is a $\text{CCS}_{g'+d-1}$. Indeed, given

$$(c_1 - c_0, \dots, c_{g'+d-1} - c_0) \in \text{Im}(\psi) \cap A^{g'+d-1,0},$$

we have

$$\sum_{i=1}^{g'+d-1} c_i = (g' + d - 1)c_0$$

so that

$$\sum_{i=1}^{g'+d-1} \{c_i\} = (g' + d - 1)\{c_0\} \in CH_0(C).$$

It follows that

$$\sum_{i=1}^{g'+d-1} \{c_i - c_0\} = (g' + d - 1)\{0_A\} \in CH_0(A).$$

\square

Since the Torelli morphism $\mathcal{M}_3 \rightarrow \mathcal{A}_3$ is dominant the previous proposition provides a one-dimensional orbit of degree 3 in a very general abelian 3-fold. Our methods do not seem to provide any way to rule out the existence of a one-parameter family of one-dimensional CCS_3 on a very general abelian 3-fold. Yet, the study of zero cycles on Jacobians does not seem to readily provide an example of such a family. This motivates the following:

Question 2.15. *Does a very general abelian 3-fold admit a one-parameter family of normalized one-dimensional orbits of degree 3?*

3. DEGENERATION AND PROJECTION

In this section we generalize Voisin's method from Section 1 of [14] to powers of abelian varieties. The key difference is that our generalization requires technical assumptions which are automatically satisfied in Voisin's setting.

Given \mathcal{A}/S a locally complete family of abelian varieties of dimension g , and a positive integer $l < g$, let $S_\lambda \subset S$ denote loci along which

$$\mathcal{A}_s \sim \mathcal{B}_s^\lambda \times \mathcal{E}_s^\lambda,$$

where $\mathcal{B}^\lambda/S_\lambda$ and $\mathcal{E}^\lambda/S_\lambda$ are locally complete families of abelian varieties of dimension l and $g-l$ respectively, and the index $\lambda \in \Lambda_l$ encodes the structure of the isogeny. Given a positive integer $l' < l$ we will also be concerned, inside each S_λ , with loci $S_{\lambda,\mu}$ along which

$$\mathcal{B}_s^\lambda \sim \mathcal{D}_s^{\lambda,\mu} \times \mathcal{F}_s^{\lambda,\mu},$$

where $\mathcal{D}^{\lambda,\mu}/S_{\lambda,\mu}$ and $\mathcal{F}^{\lambda,\mu}/S_{\lambda,\mu}$ are locally complete families of abelian varieties of dimension l' and $l-l'$ respectively, and the index $\mu \in \Lambda_{l'}^\lambda$ encodes the structure of the isogeny. For our applications we will mostly be concerned with $(l, l') = (g-1, 2)$. Upon passing to a generically finite cover of S_λ and $S_{\lambda,\mu}$ we can assume that we have projections

$$\begin{aligned} p_\lambda : \mathcal{A}_{S_\lambda}^k &\rightarrow (\mathcal{B}^\lambda)^k \\ p_\mu : (\mathcal{B}_{S_{\lambda,\mu}}^\lambda)^k &\rightarrow (\mathcal{D}^{\lambda,\mu})^k, \end{aligned}$$

and we let

$$p_{\lambda,\mu} := p_\mu \circ p_\lambda$$

for $\mu \in \Lambda_{l'}^\lambda$. Note that, to keep an already unruly notation in check, we suppress the power k from the notation of the projections. Given a subvariety $\mathcal{Z} \subset \mathcal{A}^k/S$ we consider the following subsets of S and conditions on \mathcal{Z}

$$R_{gf} = \bigcup_\lambda \{s \in S_\lambda : p_\lambda|_{\mathcal{Z}_s} : \mathcal{Z}_s \rightarrow \mathcal{B}_s^\lambda \text{ is generically finite on its image}\},$$

$$R_{ab} = \bigcup_\lambda \{s \in S_\lambda : p_\lambda(\mathcal{Z}_s) \text{ is not an abelian subvariety of } \mathcal{B}_s^\lambda\},$$

$$R_{st} = \bigcup_\lambda \{s \in S_\lambda : p_\lambda(\mathcal{Z}_s) \text{ is not stabilized by an abelian subvariety of } \mathcal{B}_s^\lambda\},$$

$$(*) \quad R_{gf} \subset S \text{ is dense,}$$

$$(**) \quad R_{ab} \cap R_{gf} \subset S \text{ is dense,}$$

$$(***) \quad R_{st} \cap R_{gf} \subset S \text{ is dense.}$$

Note that these sets and conditions depend on \mathcal{Z} and l and, while \mathcal{Z} should usually be clear from the context, we will say $(*)$ holds for a specified value of l . Moreover, we will always

assume that $\mathcal{Z} \rightarrow S$ is irreducible and has irreducible fibers. Given an abelian variety A we will denote by $T_A := T_{A,0_A}$ the tangent space to A at $0_A \in A$. We let

$$\begin{aligned}\mathcal{T}/S &:= T_{\mathcal{A}}/S, \\ G/S &:= \text{Gr}(g-l, \mathcal{T})/S, \\ G'/S &:= \text{Gr}(g-l', \mathcal{T})/S,\end{aligned}$$

and we consider the following sections

$$\begin{aligned}\sigma_\lambda : S_\lambda &\rightarrow G_{S_\lambda} = \text{Gr}(g-l, \mathcal{T}_{S_\lambda}), & \sigma_\lambda(s) &:= T_{\ker(p_{\lambda,s})}, \\ \sigma_{\lambda,\mu} : S_{\lambda,\mu} &\rightarrow G'_{S_{\lambda,\mu}} = \text{Gr}(g-l', \mathcal{T}_{S_{\lambda,\mu}}), & \sigma_{\lambda,\mu}(s) &:= T_{\ker(p_{\lambda,\mu,s})}.\end{aligned}$$

Lemma 3.1. *Let \mathcal{A}/S be a family of abelian varieties, and $\mathcal{Z} \subset \mathcal{A}$ a subvariety which is flat over the base. Then, the set of $s \in S$ such that \mathcal{Z}_s is stabilized by a positive dimensional abelian subvariety of \mathcal{A}_s is closed in S .*

Proof. Consider the morphism $\mathcal{Z} \times_S \mathcal{A} \rightarrow \mathcal{A}$ given by $(z, a) \mapsto (z+a)$ and let \mathcal{R} be the preimage of $\mathcal{Z} \subset \mathcal{A}$. Since $\mathcal{Z} \rightarrow S$ is flat, so is $\mathcal{Z} \times_S \mathcal{A} \rightarrow \mathcal{A}$. Since flat morphisms are open, the image of $\mathcal{Z} \times_S \mathcal{A} \setminus \mathcal{R}$ in \mathcal{A} is open. The complement of this image is the closed subset $\mathcal{B} \subset \mathcal{A}$ which is the maximal abelian subscheme stabilizing \mathcal{Z} . Finally, the subset of S over which \mathcal{B} has positive dimensional fibers is closed by upper semi-continuity of fiber dimension. \square

Lemma 3.2. $\bigcup_{\lambda \in \Lambda_l} \sigma_\lambda(S_\lambda) \subset G$ is dense.

Proof. It suffices to consider the locus of abelian varieties isogenous to E^g for some elliptic curve E . This locus is dense in S and, given s in this locus and any $M \in M_{k \times (g-l)}(\mathbb{Z})$ of rank $(g-l)$, the tangent space $T_{E_M^{g-l}} \in G_s$ is contained in $\sigma_\lambda(S_\lambda)$ for some $\lambda \in \Lambda_l$. Since

$$\{T_{E_M^{g-l}} : M \in M_{k \times (g-l)}(\mathbb{Z}), \text{rank}(M) = g-l\} \subset G_s$$

is dense in G_s , the result follows. \square

In the following \mathcal{A}/S will be an almost complete family of abelian varieties.

Lemma 3.3. *If $\mathcal{Z} \subset \mathcal{A}^k$ is foliated by positive dimensional CCS_k and $\dim A \geq 2$, then, for very general $s \in S$, the subset \mathcal{Z}_s is not an abelian subvariety of the form A_M^r .*

Proof. Consider the Zariski closed sets

$$\{s \in S : \mathcal{Z}_s = (\mathcal{A}_s)_M^r\} \subset S.$$

There are countably many such sets so it suffices to show that none of them is all of S . Suppose that \mathcal{A}_M^r is foliated by positive dimensional CCS_k . By the previous lemma, there is a $\lambda \in \Lambda_2$ such that $p_\lambda((\mathcal{A}_s)_M^r) = (\mathcal{B}_s^\lambda)_M^r$ is also foliated by positive dimensional orbits for generic $s \in S_\lambda$. This contradicts Corollary 2.10. \square

Lemma 3.4. *If $\mathcal{Z} \subset \mathcal{A}^k/S$ is foliated by positive dimensional orbits and $l \geq 2$, then*

$$(*) \implies (**).$$

Proof. First, observe that if $p_\lambda|_{\mathcal{Z}_s} : \mathcal{Z}_s \rightarrow \mathcal{B}_s^k$ is generically finite on its image, then its image is foliated by positive dimensional orbits. Moreover, if $R_{gf} \cap S_\lambda \neq \emptyset$, then $R_{gf} \cap S_\lambda$ is open in S_λ . Let λ be such that $R_{gf} \cap S_\lambda \neq \emptyset$. For very general $s \in R_{gf} \cap S_\lambda$ the abelian variety \mathcal{B}_s is simple. Thus, if the Zariski closed subset $p_\lambda(\mathcal{Z}_s)$ is an abelian subvariety of \mathcal{B}_s^k , it

must be abelian subvariety of the form $\mathcal{B}_{M,t}^r$, contradicting Lemma 2.10. Hence if $R_{gf} \cap S_\lambda$ is non-empty then $R_{gf} \cap R_{ab} \cap S_\lambda$ is dense in S_λ . \square

We will also need the following Zariski closed subsets

$$S_\lambda(B) = \{s \in S_\lambda : \mathcal{B}_s^\lambda \cong B\} \subset S_\lambda$$

$$S_{\lambda,\mu}(D, F) = \{s \in S_{\lambda,\mu} : \mathcal{D}_s^{\lambda,\mu} \cong D, \mathcal{F}_s^{\lambda,\mu} \cong F\} \subset S_{\lambda,\mu}.$$

The main result we will prove in this section is a generalization of Voisin's method from [14]. Recall that, given varieties X, S , a variety $\mathcal{Z} \subset X_S$ dominant over S gives rise to a morphism from (an open in) S to the Chow variety parametrizing cycles of class $[\mathcal{Z}_s]$ on X , where $s \in S$ is generic. We remind the reader of the notational convention of Remark 2.1, which allows us to remove the words in parenthesis from the previous sentence.

Proposition 3.5. *Let $\mathcal{Z} \subset \mathcal{A}^k$ be a d -dimensional variety dominant over S , and satisfying (*) and (**). Then there exists a $\lambda \in \Lambda_l$ such that*

$$p_\lambda(\mathcal{A}_{S_\lambda(B)}) \subset B_{S_\lambda(B)} = \mathcal{B}_{S_\lambda(B)}^\lambda$$

gives rise to a finite morphism from $S_\lambda(B)$ to the appropriate Chow variety.

From Lemma 3.4, if \mathcal{Z} is foliated by CCS_k then it satisfies (**). Hence, the key assumption for our applications will be (*). We propose to use these methods to prove Conjecture 1.3 in the following way: Assume that a very general abelian variety of dimension $2k - 1$ has a one-dimensional CCS_k . This gives $\mathcal{Z} \subset \mathcal{A}^k/S$ of relative dimension 1. It is easy to show that (*) holds in this setting so we can use the previous proposition to get a one-parameter family of one-dimensional CCS_k

$$p_\lambda(\mathcal{Z}_{S_\lambda(B)}) \subset (B_{S_\lambda(B)})^k$$

in a generic abelian variety B of dimension $(g - 1)$ with some polarization θ^λ . This gives

$$\mathcal{Z}' \subset (\mathcal{B}^\lambda)^k/S_\lambda$$

which has relative dimension 2 and such that \mathcal{Z}'_s is foliated by positive dimensional orbits for generic $s \in S_\lambda$.

We can hope to inductively apply Proposition 3.5 to \mathcal{Z}'/S_λ , eventually getting a large-dimensional subvariety of B^k foliated by positive dimensional orbits, for an abelian surface B . Corollary 2.12 would then provide the desired contradiction. The key issue here is to ensure that condition (*) is satisfied. At each step the dimension of the variety \mathcal{Z} to which we apply Proposition 3.5 grows. Hence, it gets harder and harder to ensure generic finiteness of the projection. We have found a way around this using the fact that the variety \mathcal{Z} to which we apply the proposition is obtained by successive degenerations and projections. We will introduce this argument in the following section.

Proof. We first reduce to the case where the condition (***) holds. Consider $s_0 \in R_{gf} \cap R_{st}$ such that $\mathcal{B}_{s_0}^{\lambda_0}$ is simple, where $s_0 \in S_{\lambda_0}$ and $p_{\lambda_0}|_{\mathcal{Z}_{s_0}}$ is generically finite on its image. Suppose that $p_{\lambda_0}(\mathcal{Z}_0)$ is stabilized by $(\mathcal{B}_{s_0}^{\lambda_0})_M^r$ but not by any larger abelian subvariety of $\mathcal{B}_{s_0}^{\lambda_0}$. Then

$$p_{\lambda_0}(\mathcal{Z}_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r \subset (\mathcal{B}_{s_0}^{\lambda_0})^k/(\mathcal{B}_{s_0}^{\lambda_0})_M^r$$

is not stabilized by any abelian subvariety of $(\mathcal{B}_{s_0}^{\lambda_0})^k/(\mathcal{B}_{s_0}^{\lambda_0})_M^r$. We have a diagram

$$(3.1) \quad \begin{array}{ccc} \mathcal{Z}_{S_{\lambda_0}}/(\mathcal{A}_{S_{\lambda_0}})_M^r & \overset{g}{\dashrightarrow} & \mathrm{Gr}(d, \mathcal{T}_{S_{\lambda_0}}^k/T_{(\mathcal{A}_{S_{\lambda_0}})_M^r}) \\ p_{\lambda_0} \downarrow & & \downarrow \pi_{\sigma_{\lambda_0}(S_{\lambda_0})} \\ p_{\lambda_0}(\mathcal{Z}_{S_{\lambda_0}})/(\mathcal{B}_{S_{\lambda_0}})_M^r & \overset{g}{\dashrightarrow} & \mathrm{Gr}\left(d, \mathrm{Gr}(d, \mathcal{T}_{S_{\lambda_0}}^k/[\sigma_{\lambda_0}(S_{\lambda_0})^k + T_{(\mathcal{B}^{\lambda_0})_M^r}])\right), \end{array}$$

where g denotes the Gauss map. Here $\pi_{\sigma_{\lambda_0}(S_{\lambda_0})}$ is the map induced by the quotient

$$\mathcal{T}_{S_{\lambda_0}}^k/T_{(\mathcal{A}_{S_{\lambda_0}})_M^r} \rightarrow \mathcal{T}_{S_{\lambda_0}}^k/[\sigma_{\lambda_0}(S_{\lambda_0})^k + T_{(\mathcal{B}^{\lambda_0})_M^r}].$$

We also denote by p_{λ_0} the map induced by p_{λ_0} on $\mathcal{Z}_{S_{\lambda_0}}/(\mathcal{A}_{S_{\lambda_0}})_M^r$. Now, consider the base change by $G \rightarrow S$

$$\mathcal{Z}_G \subset \mathcal{A}_G^k.$$

The section σ_{λ_0} is a closed immersion $S_{\lambda_0} \rightarrow G$ and $\mathcal{Z}_{S_{\lambda_0}}$ is the base change of \mathcal{Z}_G under this immersion. The upper right corner of diagram (3.1) is the base-change by $\sigma_{\lambda_0} : S_{\lambda_0} \rightarrow G$ of the diagram

$$(3.2) \quad \begin{array}{ccc} \mathcal{Z}_G/(\mathcal{A}_M^r)_G & \overset{g}{\dashrightarrow} & \mathrm{Gr}(d, \mathcal{T}_G^k/T_{(\mathcal{A}_M^r)_G}) \\ & & \downarrow \pi \\ & & \mathrm{Gr}(d, [\mathcal{T}_G/\mathcal{U}]^k/T_{(\mathcal{B}_M^r)_G}), \end{array}$$

where $\mathcal{U} \rightarrow G := \mathrm{Gr}(g-l, T)$ is the universal bundle and π is the rational map induced by the quotient map. The composition $\pi \circ g$ is defined on a Zariski open in \mathcal{Z}_G which meets $\mathcal{Z}_{\sigma_{\lambda_0}(s_0)}/\mathcal{A}_{M, \sigma_{\lambda_0}(s_0)}^r$ non-trivially. Indeed g is defined on the smooth locus of

$$\mathcal{Z}_{\sigma_{\lambda_0}(s_0)}/(\mathcal{A}_{\sigma_{\lambda_0}(s_0)})_M^r,$$

and the restriction of p_{λ_0} to $\mathcal{Z}_{\sigma_{\lambda_0}(s_0)}/(\mathcal{A}_{\sigma_{\lambda_0}(s_0)})_M^r$ is generically finite on its image by the following:

Lemma 3.6. *Consider an abelian variety $A \sim B \times E$, where B and E are abelian varieties of smaller dimension, and let p be the projection $A^k \rightarrow B^k$. If $Z \subset A^k$ is such that $p|_Z : Z \rightarrow p(Z)$ is generically finite, then the projection $p : A^k/A_M^r \rightarrow B^k/B_M^r$ is such that $p|_{Z/A_M^r}$ is generically finite on its image.*

Proof. Since $p|_{Z/A_M^r}$ is proper and locally of finite presentation, it suffice to show that it is quasi-finite. The fiber of $A^k \rightarrow B^k/B_M^r$ over $p(z) \in p(Z)/B_M^r$ for $z \in Z$ is the set of all A_M^r -translates of the fiber of p over $p(z)$. Hence the fiber of $p|_{Z/A_M^r}$ over $p(z)$ is finite. \square

We deduce that $q := g \circ \pi$ is defined in an open in the smooth locus of $\mathcal{Z}_{s_0}/(\mathcal{A}_{M, s_0}^r)$, so that q is defined in an open in \mathcal{Z}_G meeting $\mathcal{Z}_{\sigma_{\lambda_0}(s_0)}/(\mathcal{A}_{s_0})_M^r$ non-trivially.

Since $p_{\lambda_0}(\mathcal{Z}_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r$ is not stabilized by any abelian subvariety of $(\mathcal{B}_{s_0}^{\lambda_0})^k$, the Gauss map g at the bottom of diagram (3.1) is defined on the smooth locus of $p_{\lambda_0}(\mathcal{Z}_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r$ and generically finite by results of Griffiths and Harris (see (4.14) in [5]). The fact that the

restriction of p_{λ_0} to $\mathcal{Z}_{s_0}/(\mathcal{A}_{s_0})_M^r$ is generically finite on its image implies that the following composition is generically finite on its image

$$g \circ p_{\lambda_0} : \mathcal{Z}_{s_0}/(\mathcal{A}_{s_0})_M^r \dashrightarrow \text{Gr}\left(d, \mathcal{F}^k/[\sigma_{\lambda_0}(s_0)^k + T_{(\mathcal{B}_{s_0}^{\lambda_0})_M^r}]\right).$$

It follows that, restricting to an open of G if needed, the map $q_t := (\pi \circ g)_t$ is defined and is generically finite on its image for all $t \in G$. This generic finiteness statement implies that, for $s \in S_\lambda$ (such that $\sigma_\lambda(s)$ lies in an appropriate open in G), the Gauss map of $p_\lambda(\mathcal{Z}_s)/(\mathcal{B}_s^\lambda)_M^r$ is generically finite. Namely, for such an s , the variety $p_\lambda(\mathcal{Z}_s/(\mathcal{B}_s^\lambda)_M^r)$ is not stabilized by an abelian subvariety. Moreover, if $\lambda \in \Lambda_t$ and B in the family \mathcal{B}^λ are such that the family

$$p_\lambda(\mathcal{Z}_{S_\lambda(B)}/\mathcal{A}_M^r) \subset (B^k/B_M^r)_{S_\lambda(B)}$$

gives rise to a finite morphism from $S_\lambda(B)$ to the appropriate Chow variety of B^k/B_M^r , then the family

$$p_\lambda(\mathcal{Z}_{S_\lambda(B)}) \subset B_{S_\lambda(B)}^k$$

also gives rise to a finite morphism from $S_\lambda(B)$ to the appropriate Chow variety of B^k . Hence, replacing \mathcal{Z} by $\mathcal{Z}/\mathcal{A}_M^r$ and \mathcal{A}^k by $\mathcal{A}^k/\mathcal{A}_M^r$ we are reduced to the case where $(***)$ holds.

Now consider the analogue of diagram (3.1)

$$(3.3) \quad \begin{array}{ccc} \mathcal{Z}_{S_\lambda} & \dashrightarrow^g & \text{Gr}(d, T^k) \\ p_\lambda \downarrow & & \downarrow \pi_{\sigma_\lambda(S_\lambda)} \\ p_\lambda(\mathcal{Z}_{S_\lambda}) & \dashrightarrow^g & \text{Gr}(d, [T/\mathcal{U}_{\sigma_\lambda(S_\lambda)}]^k) \end{array}$$

as well as the analogue of diagram (3.2)

$$(3.4) \quad \begin{array}{ccc} \mathcal{Z}_G & \dashrightarrow^g & \text{Gr}(d, T_G^k) \\ & & \downarrow \pi \\ & & \text{Gr}(d, [T_G/\mathcal{U}]^k). \end{array}$$

By the discussion above there is an open in \mathcal{Z}_G on which the composition $q := \pi \circ g$ is defined and generically finite on its image. The diagram (3.3) provides a factorization of q along each $\sigma_\lambda(S_\lambda)$

$$(3.5) \quad \begin{array}{ccc} \mathcal{Z}_{\sigma_\lambda(S_\lambda)} \cong \mathcal{Z}_{S_\lambda} & \dashrightarrow^{q:=\pi \circ g} & \text{Gr}(d, [T_G/\mathcal{U}]^k) \\ & \searrow p_\lambda & \nearrow g \\ & p_\lambda(\mathcal{Z}_{S_\lambda}) & \end{array}$$

Hence (3.5) gives a factorization of q on a Zariski dense subset of the base G .

Lemma 3.7. *Let \mathcal{Z}/S be a family with irreducible fibers and base, and $q : \mathcal{Z}/S \rightarrow \mathcal{X}/S$ be such that $q_s : \mathcal{Z}_s \rightarrow \mathcal{X}_s$ is generically finite for each $s \in S$. Consider $S' \subset S$, a Zariski dense subset, and suppose that for each $s' \in S'$ we have a factorization of $q_{s'}$*

$$\mathcal{Z}_s \xrightarrow{f_{s'}} f_{s'}(\mathcal{Z}_s) \xrightarrow{g_{s'}} \mathcal{X}_s.$$

Then there is a family \mathcal{Z}'/S , morphisms $p : \mathcal{Z} \rightarrow \mathcal{Z}'$ and $p' : \mathcal{Z}' \rightarrow \mathcal{X}$, and a Zariski dense subset $S'' \subset S'$ such that, for any $s'' \in S''$, the morphisms $p_{s''}(\mathcal{Z}_{s''})$ and $f(\mathcal{Z}_{s''})$ are birational, and $p_{s''}$ and $p'_{s''}$ induce the same morphism on function fields as $f_{s''}$ and $g_{s''}$ respectively.

Proof. Restrict to a Zariski open subset of \mathcal{X} (which we call \mathcal{X} in keeping with remark 2.1) over which q is finite étale and such that $\mathcal{X} \rightarrow S$ is smooth. By work of Hironaka, we can find a compactification $\overline{\mathcal{X}}$ of \mathcal{X} with simple normal crossing divisors at infinity. Restricting to an open in the base, we can assume that we have $\mathcal{X}/S \subset \overline{\mathcal{X}}/S$, such that $\overline{\mathcal{X}}_s \setminus \mathcal{X}_s$ is an snc divisor for any $s \in S$. One can use a version of Ehresmann's lemma allowing for an snc divisor at infinity to see that $\mathcal{X} \rightarrow S$ is a locally-trivial fibration in the category of smooth manifolds.

It follows that we get a diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & \mathcal{X} \\ & & \downarrow \\ & & S, \end{array}$$

where q is a covering. Note that we renamed as \mathcal{Z} an open subset of \mathcal{Z} over which q is étale. To complete the proof of Proposition 3.5 we will need the following Lemma.

Lemma 3.8. *Given a diagram as above with \mathcal{Z}_s connected for every s , and a factorization*

$$\begin{array}{ccc} \mathcal{Z}_{s_0} & \xrightarrow{q} & \mathcal{X}_{s_0} \\ & \searrow f_{s_0} & \nearrow \\ & f_{s_0}(\mathcal{Z}_{s_0}) & \end{array}$$

there is a factorization

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & \mathcal{X} \\ & \searrow f & \nearrow \\ & f(\mathcal{Z}) & \end{array}$$

which identifies with the original factorization over $s_0 \in S$.

Proof. Consider the Galois closure $\mathcal{Z}' \rightarrow \mathcal{X}$ of the covering $q : \mathcal{Z} \rightarrow \mathcal{X}$. Note that \mathcal{Z}'_{s_0} is connected. Indeed there is a normal subgroup of the Galois group of \mathcal{Z}'/\mathcal{X} corresponding to a deck transformation inducing the trivial permutation of the connected components of \mathcal{Z}'_{s_0} . This subgroup corresponds to a cover above \mathcal{Z} since \mathcal{Z}_{s_0} is connected. It follows that the map $\text{Gal}(\mathcal{Z}'/\mathcal{X}) \rightarrow \text{Gal}(\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0})$ is injective because a deck transformation which is the identity on the base and on fibers must be the identity. Since $\text{Gal}(\mathcal{Z}'/\mathcal{X})$ has order

$$d := \deg(\mathcal{Z}'/\mathcal{X}) = \deg(\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0}),$$

and $\text{Gal}(\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0})$ has order at most d , this restriction morphism must be an isomorphism and $\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0}$ is thus also Galois. One then has an equivalence of categories between the

poset of intermediate coverings of \mathcal{Z}'/\mathcal{X} and that of $\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0}$, and hence between the poset of intermediate coverings of \mathcal{Z}/\mathcal{X} and that of $\mathcal{Z}_{s_0}/\mathcal{X}_{s_0}$. \square

By the previous lemma, to each factorization $f_{s'}$ we can uniquely associate an intermediate cover $\mathcal{Z} \rightarrow \mathcal{Z}^{s'}$ of $\mathcal{Z} \rightarrow \mathcal{X}$, which agrees with $f_{s'}$ at s' . Since there are only finitely many intermediate covers of $\mathcal{Z} \rightarrow \mathcal{X}$, we get a partition of S' according to the isomorphism type of the cover $\mathcal{Z} \rightarrow \mathcal{Z}^{s'}$. One subset $S'' \subset S'$ of this partition must be dense in S . Let $f : \mathcal{Z} \rightarrow f(\mathcal{Z})$ be the corresponding intermediate cover. \square

For the rest of the proof of Proposition 3.5 we are back in the situation of diagrams (3.3), (3.4), and (3.5).

Corollary 3.9. *There is a variety \mathcal{Z}'/G , a morphism $p : \mathcal{Z} \rightarrow \mathcal{Z}'$, and a subset $\Lambda_{l,0} \subset \Lambda_l$ such that*

$$\bigcup_{\lambda \in \Lambda_{l,0}} \sigma_\lambda(S_\lambda) \subset G$$

is dense, and such that $p_\lambda(\mathcal{Z}_t)$ and $p(\mathcal{Z}_t)$ are birational, and $p_t : \mathcal{Z}_t \rightarrow p(\mathcal{Z}_t)$ and $p_{\lambda,t} : \mathcal{Z}_t \rightarrow p_\lambda(\mathcal{Z}_t)$ induce the same map on function fields, for any $\lambda \in \Lambda_{l,0}$, $t \in \sigma_\lambda(S_\lambda)$.

Proof. This follows from the previous lemma and its proof once we observe that the intermediate covering of q (or rather of an appropriate finite étale restriction of q as above) associated to the factorization $p_{\lambda,t} : \mathcal{Z}_t \rightarrow p_\lambda(\mathcal{Z}_t)$ is independent of $t \in \sigma_\lambda(S_\lambda)$. \square

Remark 3.10. A technical point is that the maps p_λ are a priori only defined after passing to a generically finite cover of S_λ . This does not cause problem as $p_{\lambda,s}$ is defined without passing to a generically finite cover and, given a generically finite cover $S'_\lambda \rightarrow S_\lambda$, the isomorphism type of the covering $p_{\lambda,s} : \mathcal{Z}_s \rightarrow p_{\lambda,s}(\mathcal{Z}_s)$ is the same as that of $p_{\lambda,s} : \mathcal{Z}'_s \rightarrow p_{\lambda,s}(\mathcal{Z}'_s)$, where $\mathcal{Z}'^\#$ is obtained from the cover as prescribed in Remark 2.1.

To finish the proof of Proposition 3.5, consider desingularizations

$$\tilde{p} : \tilde{\mathcal{Z}}/G \rightarrow \tilde{\mathcal{Z}}'/G$$

with smooth fibers over G . We have the inclusion

$$j : \mathcal{Z}/G \rightarrow \mathcal{A}^k/G$$

as well as

$$\tilde{j} : \tilde{\mathcal{Z}}/G \rightarrow \mathcal{A}^k/G.$$

The morphism \tilde{j} gives rise to a pullback map

$$\tilde{j}^* : \text{Pic}^0(\mathcal{A}^k/G) \rightarrow \text{Pic}^0(\tilde{\mathcal{Z}}/G).$$

Since \tilde{p} is generically finite on fibers we can consider the composition

$$\tilde{p}_* \circ \tilde{j}^* : \text{Pic}^0(\mathcal{A}^k/G) \rightarrow \text{Pic}^0(\tilde{\mathcal{Z}}/G) \rightarrow \text{Pic}^0(\tilde{\mathcal{Z}}'/G).$$

This is a morphism of abelian schemes and we will show that it is non-zero along $\sigma_\lambda(S_\lambda)$, $\lambda \in \Lambda_{l,0}$. Consider $t \in \sigma_\lambda(S_\lambda)$. Then \mathcal{A}_t is isogenous to $\mathcal{B}_t^\lambda \times \mathcal{E}_t^\lambda$. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_t & \xrightarrow{j} & \mathcal{A}_t^k \\ p_{\lambda|\mathcal{Z}_t} \downarrow & & \downarrow p_\lambda \\ p_\lambda(\mathcal{Z}_t) & \xrightarrow{j'} & (\mathcal{B}_t^\lambda)^k. \end{array}$$

Consider a desingularization of $p_\lambda(\mathcal{Z}_t)$ and the induced map

$$\tilde{j}' : \widetilde{p_\lambda(\mathcal{Z}_t)} \rightarrow (\mathcal{B}_t^\lambda)^k.$$

Using the fact that \tilde{p}_t identifies birationally to $p_{\lambda,t}$, we get the following commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0(\tilde{\mathcal{Z}}_t) & \xleftarrow{\tilde{j}^*} & \mathrm{Pic}^0(\mathcal{A}_t^k) \\ \tilde{p}^* \uparrow & & \uparrow p_\lambda^* \\ \mathrm{Pic}^0(\tilde{\mathcal{Z}}'_t) \cong \mathrm{Pic}^0(\widetilde{p_\lambda(\mathcal{Z}_t)}) & \xleftarrow{\tilde{j}'^*} & \mathrm{Pic}^0(\mathcal{B}_t^\lambda)^k. \end{array}$$

It follows that

$$\tilde{p}_* \circ (\tilde{j}^* \circ p_\lambda^*) = \tilde{p}_* \circ (\tilde{p}^* \circ \tilde{j}'^*) = (\tilde{p}_* \circ \tilde{p}^*) \circ \tilde{j}'^* = [\mathrm{deg}(\tilde{p})] \circ \tilde{j}'.$$

Since $p_\lambda(\mathcal{Z}_t)$ is positive dimensional, the morphism \tilde{j}' is non-zero and so $\tilde{p}_* \circ \tilde{j}^*$ is non-zero.

Hence the kernel of $\tilde{p}_* \circ \tilde{j}^*$ is an abelian subscheme of \mathcal{A}^k which is not all of \mathcal{A}^k . For very general $t \in G$ the abelian variety \mathcal{A}_t is simple. Therefore, for such a t the abelian subvariety $\ker(\tilde{p}_* \circ \tilde{j}^*)_t$ is of the form $(\mathcal{A}_t)_M^r$, with $M \in M_{k \times r}(\mathbb{Z})$ of rank r , and $r \leq k-1$. Choosing M and r such that

$$\{t \in G : \ker(\tilde{p}_* \circ \tilde{j}^*)_t = (\mathcal{A}_t)_M^r\} \subset G$$

is dense, and observing that this set is closed, we see that $\ker(\tilde{p}_* \circ \tilde{j}^*)_t = (\mathcal{A}_t)_M^r$ for all $t \in G$.

Note that, for $t \in \sigma_\lambda(S_\lambda)$, we have

$$\mathrm{Pic}^0(\ker(p_{\lambda,t})) / \ker(\tilde{p}_* \circ \tilde{j}^*)_t \cap \ker(p_{\lambda,t}) = \mathrm{Pic}^0(\ker(p_{\lambda,t})) / (\mathcal{A}_t)_M^r \cap \ker(p_{\lambda,t}) \neq 0.$$

Now consider $\lambda \in \Lambda_l$ such that $\sigma_\lambda(S_\lambda) \neq \emptyset$, namely such that some point in $\sigma_\lambda(S_\lambda)$ has survived our various restriction to Zariski open subsets, and $B \in \mathcal{B}^\lambda$ such that $\sigma_\lambda(S_\lambda(B)) \neq \emptyset$. Suppose that there is a curve $C \subset \sigma_\lambda(S_\lambda(B)) \cong S_\lambda(B)$ such that $p_\lambda(\tilde{Z}_t) = p_\lambda(\tilde{Z}_{t'})$ for any $t, t' \in C$, namely such that C is contracted by the morphism from $S_\lambda(B)$ to the Chow variety associated to the family $p_\lambda(\mathcal{Z}_{S_\lambda(B)}) \subset B_{S_\lambda(B)}^k$. Now, since $\mathrm{Pic}^0(\tilde{Z}'_t) \cong \mathrm{Pic}^0(\widetilde{p_\lambda(\mathcal{Z}_t)})$ does not depend on $t \in C$, it must contain a variable abelian variety. This provides the desired contradiction and completes the proof of Proposition 3.5. \square

4. SALVAGING GENERIC FINITENESS AND A PROOF OF VOISIN'S CONJECTURE

In this section we refine the results from the previous section in order to bypass assumption (*) in the inductive application of Proposition 3.5. The idea is quite simple: In the last section we saw that we can degenerate to abelian varieties A isogenous to $B \times E$ in such a way that, if we consider the restriction of the projection $A^k \rightarrow B^k$ to $Z \subset A^k$, the image of this projection varies with E . Here we want to degenerate to abelian varieties isogenous to $D \times F \times E$, where E is an elliptic curve, and consider the restriction of the projections $A^k \rightarrow D^k$ and $A^k \rightarrow (D \times F)^k$ to $Z \subset A^k$. We can do this in such a way that the images of both of these projections vary with E . Hence, if we consider in $(D \times F)^k$ and D^k the union of the image of these projections for every E , we get varieties $Z_1 \subset (D \times F)^k$ and $Z_2 \subset D^k$ of dimension $\dim Z + 1$, and the restriction of the projection $(D \times F)^k \rightarrow D^k$ to Z_1 has image Z_2 . It follows at once that this restriction is generically finite on its image. We spend this section making this simple idea rigorous and deducing a proof of Theorem 4.4.

The families $\mathcal{B}^\lambda/S_\lambda$, $\lambda \in \Lambda_{(g-1)}$ introduced in the last section are families of $(g-1)$ -dimensional abelian varieties with some polarization type θ^λ . Hence, they give rise to a diagram

$$(4.1) \quad \begin{array}{ccc} (\mathcal{B}^\lambda)^k & \xrightarrow{\varphi_\lambda} & \mathcal{A}'^k \\ \downarrow & & \downarrow \\ S_\lambda & \xrightarrow{\psi_\lambda} & S' \end{array}$$

where \mathcal{A}'/S' is the universal family over the moduli stack of abelian varieties of dimension $(g-1)$ with polarization θ^λ . Note that \mathcal{A}'/S' depends of course on λ but we suppress λ from the notation. We will think of \mathcal{A}'/S' as another locally complete family of abelian varieties. This family comes with its own set $\Lambda'_{l'}$ indexing loci $S_{\eta'}$ along which $\mathcal{A}'_s \sim \mathcal{B}'^{\eta'}_s \times \mathcal{E}'^{\eta'}_s$. Let $\varphi_{\lambda,\mu}$ be the composition of $\varphi_\lambda|_{(\mathcal{B}^\lambda_{S_{\lambda,\mu}})^k}$ with

$$(\mathcal{D}^{\lambda,\mu})^k \rightarrow (\mathcal{D}^{\lambda,\mu} \times \mathcal{F}^{\lambda,\mu})^k \rightarrow (\mathcal{B}^\lambda_{S_{\lambda,\mu}})^k,$$

where the last map is the isogeny encoded by μ . We get a diagram

$$(4.2) \quad \begin{array}{ccc} (\mathcal{D}^{\lambda,\mu})^k & \xrightarrow{\varphi_{\lambda,\mu}} & (\mathcal{B}'^\eta)^k \\ \downarrow & & \downarrow \\ S_{\lambda,\mu} & \longrightarrow & S'_\eta \end{array}$$

where η is some index in $\Lambda'_{l'}$, and $\psi_\lambda(S_{\lambda,\mu}) = S'_\eta$. The main result of this section is:

Proposition 4.1. *Suppose that $\mathcal{Z} \subset \mathcal{A}^k$ satisfies $(*)$ and $(**)$ for $l' \geq 2$. Then there exists a $\lambda \in \Lambda_{(g-1)}$ such that $\varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda}))/S'$ satisfies $(*)$ for l' and has relative dimension $\dim_S \mathcal{Z} + 1$.*

Proof. By Proposition 3.4, there is a $\lambda \in \Lambda_{(g-1)}$ such that $p_\lambda(\mathcal{Z}_t)$ varies with $t \in S_\lambda(B)$, for generic B in the family \mathcal{B}^λ (alternatively such that $\dim_{S'} \varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda})) = \dim_S \mathcal{Z} + 1$). The idea is to show that there is a subset $\Lambda'_{l',0} \subset \Lambda'_{l'}$ such that

$$(4.3) \quad \bigcup_{\mu \in \Lambda'_{l',0}} S_{\lambda,\mu} \subset S_\lambda \text{ is dense}$$

and such that $p_{\lambda,\mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda,\mu}(D, F)$, for generic D, F in the families $\mathcal{D}^{\lambda,\mu}, \mathcal{F}^{\lambda,\mu}$, and $\mu \in \Lambda'_{l',0}$. Indeed, we have a commutative diagram

$$\begin{array}{ccccc} & & (\mathcal{B}^\lambda_{S_{\lambda,\mu}})^k & \xrightarrow{\varphi_\lambda} & \mathcal{A}'^k \\ & \swarrow p_{\lambda,\mu} & \downarrow \varphi_{\lambda,\mu} & & \swarrow p_\eta \\ (\mathcal{D}^{\lambda,\mu})^k & \xrightarrow{\quad} & (\mathcal{B}'^\eta)^k & & \mathcal{A}'^k \\ \downarrow & & \downarrow & & \downarrow \\ & & S_{\lambda,\mu} & \xrightarrow{\quad} & S'_\eta \\ \downarrow & \parallel & \downarrow & & \downarrow \\ S_{\lambda,\mu} & \xrightarrow{\quad} & S'_\eta & & S'_\eta \end{array}$$

where $S'_\eta \subset S'$ is the image of $S_{\lambda,\mu} \rightarrow S_\lambda \rightarrow S'$. Consider the restriction of this diagram to $S_{\lambda,\mu}(D, F)$, for D and F in the families $\mathcal{D}^{\lambda,\mu}$ and $\mathcal{F}^{\lambda,\mu}$, namely

$$\begin{array}{ccccc}
& & (\mathcal{B}_{S_{\lambda,\mu}(D,F)}^\lambda)^k & \xrightarrow{\varphi_\lambda} & \mathcal{A}'^k \\
& \swarrow p_{\lambda,\mu} & \downarrow \varphi_{\lambda,\mu} & & \swarrow p_\eta \\
(\mathcal{D}_{S_{\lambda,\mu}(D,F)}^{\lambda,\mu})^k & \xrightarrow{\quad} & (\mathcal{B}'^\eta)^k & & \\
\downarrow & & \downarrow & & \downarrow \\
& & S_{\lambda,\mu}(D, F) & \xrightarrow{\quad} & S'_\eta \\
\downarrow & \parallel & \downarrow & & \parallel \\
S_{\lambda,\mu}(D, F) & \xrightarrow{\quad} & & & S'_\eta.
\end{array}$$

Now, if $p_{\lambda,\mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda,\mu}(D, F)$, we have

$$\dim \varphi_\lambda(p_\lambda(\mathcal{Z}_{S_{\lambda,\mu}(D,F)})) = \dim \varphi_{\lambda,\mu}(p_{\lambda,\mu}(\mathcal{Z}_{S_{\lambda,\mu}(D,F)})) = \dim_S \mathcal{Z} + 1,$$

so that p_η is generically finite. Hence, if there is a subset $\Lambda_{l',0}^\lambda$ such that

$$\bigcup_{\mu \in \Lambda_{l',0}^\lambda} S_{\lambda,\mu} \subset S_\lambda$$

is dense, and $p_{\lambda,\mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda,\mu}(D, F)$ for $\mu \in \Lambda_{l',0}^\lambda$, then $\varphi_\lambda(\mathcal{Z}_{S_\lambda}) \subset \mathcal{A}'^k$ satisfies condition (*) for l' .

Consider

$$R'_{st} := \bigcup_{\lambda,\mu} \{s \in S_{\lambda,\mu} : p_{\lambda,\mu}(\mathcal{Z}_s) \text{ is not stabilized by an abelian subvariety}\}.$$

Following the same argument as in the proof of Proposition 3.5, we can assume that R'_{st} is dense in S (this is the analogue of condition (***)). Let \mathcal{U}/G and \mathcal{U}'/G' be the universal families over $G := \mathbb{P}(T)$ and $G' := \text{Gr}(g - l', T)$. Consider the following diagrams analogous to (3.3) and (3.4):

$$\begin{array}{ccc}
\mathcal{Z}_{S_{\lambda,\mu}} & \overset{g}{\dashrightarrow} & \text{Gr}(d, \mathcal{T}_{S_{\lambda,\mu}}^k) \\
\downarrow p_\lambda & & \downarrow \pi_{\sigma_\lambda(S_{\lambda,\mu})} \\
(4.4) \quad p_\lambda(\mathcal{Z}_{S_{\lambda,\mu}}) & \overset{g}{\dashrightarrow} & \text{Gr}(d, [\mathcal{T}_{S_{\lambda,\mu}}/\mathcal{U}_{\sigma_\lambda(S_{\lambda,\mu})}]^k) \\
\downarrow p_\mu & & \downarrow \pi_{\sigma_{\lambda,\mu}(S_{\lambda,\mu})} \\
p_{\lambda,\mu}(\mathcal{Z}_{S_{\lambda,\mu}}) & \overset{g}{\dashrightarrow} & \text{Gr}(d, [\mathcal{T}_{S_{\lambda,\mu}}/\mathcal{U}'_{\sigma_{\lambda,\mu}(S_{\lambda,\mu})}]^k),
\end{array}$$

$$\begin{array}{ccc}
\mathcal{Z}_{G'} & \overset{g}{\dashrightarrow} & \text{Gr}(d, T_{G'}^k) \\
& & \downarrow \pi \\
(4.5) & & \text{Gr}(d, [T_{G'}/\mathcal{U}']^k).
\end{array}$$

Just as in the proof of Proposition 3.5, since \mathcal{Z} satisfies (*) and (**) for l' , we can assume that $q_t := \pi_t \circ g_t$ is generically finite on its image for any $t \in G'$, restricting to a Zariski open in G' if necessary. Note that, for $t \in S_{\lambda, \mu}$, we have a factorization

$$(4.6) \quad \begin{array}{ccc} \mathcal{Z}_t & \overset{q}{\dashrightarrow} & \text{Gr}(d, [T_{G'}/\mathcal{U}']^k) \\ & \searrow^{p_\lambda} & \nearrow^g \\ & p_\lambda(\mathcal{Z}_t) & \xrightarrow{p_\mu} p_{\lambda, \mu}(\mathcal{Z}_t) \end{array}$$

By Proposition 3.5, there is a λ such that $\dim_{S'} \varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda}))/S' = \dim_S \mathcal{Z} + 1$. One can consider analogues of Lemma 3.7 and 3.8 and see that there is a partition of $\Lambda_{l'}^\lambda$ in finitely many sets according to the isomorphism type of the covering p_μ . Hence, since

$$\bigcup_{\mu \in \Lambda_{l'}^\lambda} S_{\lambda, \mu} \subset S_\lambda$$

is dense, there is a subset $\Lambda_{l', 0}^\lambda \subset \Lambda_{l'}^\lambda$, such that

$$(4.7) \quad \bigcup_{\mu \in \Lambda_{l', 0}^\lambda} S_{\lambda, \mu} \subset S_\lambda$$

is dense, and such that $p_{\lambda, \mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda, \mu}(D, F)$ for generic D, F in the families $\mathcal{B}^{\lambda, \mu}, \mathcal{D}^{\lambda, \mu}$, and $\mu \in \Lambda_{l', 0}^\lambda$. \square

Corollary 4.2. *Suppose that a very general abelian variety of dimension g has a positive dimensional CCS_k . Then, for a very general abelian variety A of dimension $(g - l)$ there is an $(l + 1)$ -dimensional subvariety of A^k foliated by positive dimensional CCS_k .*

Proof. Under the assumption of this corollary we get $\mathcal{Z} \subset \mathcal{A}^k/S$, a CCS_k/S , for $\mathcal{A} \rightarrow S$ a locally complete family of g -dimensional abelian varieties. Apply the previous proposition inductively. By Lemma 3.3 the condition (**) follows from (*). \square

Corollary 4.3. *Conjecture 1.3 holds: a very general abelian variety of dimension $\geq 2k - 1$ has no positive dimensional CCS_k .*

Proof. Note that any $\mathcal{Z} \subset \mathcal{A}^k$ of relative dimension d satisfies (*) for $l \geq d$ since

$$\bigcup_{\lambda \in \Lambda_l} \sigma_\lambda(S_\lambda) \subset G$$

is dense. Indeed, if V is a vector space and $W \subset V^k$ has dimension $d < \dim V$, then the restriction of the projection $V^k \rightarrow (V/H)^k$ to W is an isomorphism onto its image for a generic $H \in \text{Gr}(\dim V - d, V)$. In particular if $\mathcal{Z} \subset \mathcal{A}^k$ has relative dimension 1 then it satisfies (*) for any $1 \leq l \leq g - 1$. Hence, if a very general abelian variety of dimension $2k - 1$ has a positive dimensional CCS_k , then a very general abelian surface B will be such that $B^{k, 0}$ contains a $(2k - 2)$ -dimensional subvariety foliated by positive dimensional CCS_k . This does not hold for any abelian surface, let alone generically, by Corollary 2.12. \square

Theorem 4.4. *For $k \geq 3$, a very general abelian variety of dimension at least $2k - 2$ has no positive dimensional orbits of degree k , i.e. $\mathcal{G}(k) \leq 2k - 2$.*

Proof. Let $\mathcal{Z} \subset \mathcal{A}^{k,0}$ be a one-dimensional normalized CCS_k , where $\dim_S \mathcal{A} = 2k - 2$. By the previous corollary, there is a $\lambda \in \Lambda_2$ such that $\varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda}))/S'$ has relative dimension $2k - 3$. This was obtained by successive degenerations and projections. But the morphism from $S_\lambda(B)$ to an appropriate Chow variety of $B^{k,0}$ given by

$$s \mapsto [\varphi_\lambda(p_\lambda(\mathcal{Z}_s))]$$

is a $\binom{2k-3}{2}$ -parameter family of CCS_k on $\varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda}))$. Hence, $\varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda}))$ must be foliated by CCS_k of dimension at least 2. This contradicts Corollary 2.12. \square

Corollary 4.5. *For $k \geq 3$, a very general abelian variety of dimension $\geq 2k - 2$ has gonality at least $k + 1$. In particular Conjecture 1.3 holds.*

Theorem 4.6. *A very general abelian variety of dimension $\geq 2k - 4$ does not have a 2-dimensional CCS_k for $k \geq 4$.*

Proof. Suppose \mathcal{A}/S is a locally complete family of $(2k - 4)$ -dimensional abelian varieties with some polarization θ , and that $\mathcal{Z} \subset \mathcal{A}^k$ is a 2-dimensional CCS_k/S . Using the same argument as in the proof of Corollary 4.3, we see that \mathcal{Z} satisfies $(*)$, and thus $(**)$. We now follow the proof of Theorem 4.4. \square

Theorem 4.7. *A very general abelian variety A of dimension at least $2k + 2 - l$ does not have a positive dimensional orbit of the form $|\sum_{i=1}^{k-l} \{a_i\} + l\{0_A\}|$, i.e. $\mathcal{G}_l(k) \leq 2k + 2 - l$. Moreover, if A is a very general abelian variety of dimension at least $k + 1$ the orbit $|k\{0_A\}|$ is countable, i.e. $\mathcal{G}_k(k) \leq k + 1$.*

Proof. By the results of [14], it suffices to show that a very general abelian variety of dimension $2k + 2 - l$ has no positive dimensional orbits of the form $|\sum_{i=1}^{k-l} \{a_i\} + l\{0_A\}|$. If this were not the case, we could find $\mathcal{Z} \subset \mathcal{A}^k$, a one-dimensional CCS_k , where \mathcal{A}/S is a locally complete family of $(2k + 2 - l)$ -dimensional abelian varieties, and

$$\{0_{\mathcal{A}_s}\} \times \dots \times \{0_{\mathcal{A}_s}\} \times \mathcal{A}_s^{k-l} \cap \mathcal{Z}_s \neq \emptyset$$

for every $s \in S$. By Proposition 4.1 there is a $\lambda \in \Lambda_2$ such that $\varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda}))$ has relative dimension $2k + 1 - l$. Given a generic B in the family \mathcal{B}^λ and $\underline{b} = (a_1, \dots, a_{k-l}) \in B^{k-l}$, consider

$$S_\lambda(B, \underline{b}) := \{s \in S_\lambda(B) : \underline{b} \in \phi_\lambda(p_\lambda(\mathcal{Z}_s))\}.$$

Clearly $\varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda(B, \underline{b})}))$ is a CCS_k . In particular, $\varphi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda(B, \underline{b})}))$ is foliated by CCS_k of codimension at most $2(k - l)$. This contradicts Corollary 2.12. A similar argument shows $\mathcal{G}_k(k) \leq k + 1$. \square

5. APPLICATIONS TO OTHER MEASURES OF IRRATIONALITY

We have seen how the minimal degree of a positive dimensional orbit gives a lower bound on the gonality of a smooth projective variety and used this to provide a new lower bound on the gonality of very general abelian varieties. In this section we show how one can use results about the maximal dimension of CCS_k in order to give lower bounds on other measures of irrationality for very general abelian varieties. We finish by discussing another conjecture of Voisin from [14] and its implication for the gonality of very general abelian varieties.

Recall the definitions of some of the measures of irrationality of irreducible n -dimensional projective varieties:

$$\text{irr}(X) := \min \{ \delta > 0 : \exists \text{ degree } \delta \text{ rational covering } X \rightarrow \mathbb{P}^n \}$$

$$\text{gon}(X) := \min \{ c > 0 : \exists \text{ a non-constant morphism } C \rightarrow X, \text{ where } C \text{ has gonality } c \}.$$

Additionally, we will consider the following measure of irrationality which interpolates between the *degree of irrationality* $\text{irr}(X) = \text{irr}_n(X)$ and the *gonality* $\text{gon}(X) = \text{irr}_1(X)$.

$$\text{irr}_d(X) := \min \{ \delta : \exists \text{ a } d\text{-dimensional irreducible subvariety } Z \subset X \text{ with } \text{irr}(Z) = \delta \}.$$

The methods of the previous section can be applied to get:

Corollary 5.1. *If A is a very general abelian variety of dimension $\geq 2k - 4$ and $k \geq 4$, then*

$$\text{irr}_2(A) \geq k + 1.$$

Proof. A surface with degree of irrationality k in a smooth projective variety X provides a 2-dimensional CCS_k . The result then follows from Theorem 4.6. \square

Similarly, we can use bounds on the dimension of a CCS_k to obtain bounds on the degree of irrationality of abelian varieties. To our knowledge, the best bound currently in the literature is the Sommese bound $\text{irr}(A) \geq \dim A + 1$ (see [3] Section 4), for any abelian variety A . It is an interesting fact that this bound follows easily from Voisin's Theorem 1.5. Indeed, a dominant morphism from A to $\mathbb{P}^{\dim A}$ of degree at most $\dim A$ would provide a $(\dim A)$ -dimensional $\text{CCS}_{\dim A}$. Note that Yoshihara and Tokunaga-Yoshihara ([16],[12]) provide examples of abelian surfaces A with $\text{irr}(A) = 3$, so that the Sommese bound is tight for $\dim A = 2$. In fact, we do not know of a single example of an abelian surface A with $\dim A > 3$.

Our results allow us to show that Sommese's bound is not optimal, at least for very general abelian varieties.

Theorem 5.2. *Orbits of degree k on a very general abelian variety of dimension at least $k - 1$ have dimension at most $k - 2$, for $k \geq 4$.*

Corollary 5.3. *If A is a very general abelian variety of dimension $g \geq 3$, then*

$$\text{irr}(A) \geq g + 2.$$

Proof of Theorem 5.2. Suppose that we have \mathcal{A}/S , a locally complete family of $(k - 1)$ -dimensional abelian varieties, and $\mathcal{Z} \subset \mathcal{A}^{k,0}/S$, a $(k - 1)$ -dimensional CCS_k/S . We claim that \mathcal{Z} satisfies (*) for $l = k - 2$. Assuming this, for appropriate $\lambda \in \Lambda_{(k-2)}$ and B in the family \mathcal{B}^λ , the subvariety $\phi_\lambda(p_\lambda(\mathcal{Z}_{S_\lambda(B)})) \subset B^{k,0}$ is k -dimensional and foliated by $(k - 1)$ -dimensional CCS_k . This contradicts Corollary 2.6.

To show that \mathcal{Z} satisfies (*) for $l = k - 2$, we will need the following easy lemma which we give without proof:

Lemma 5.4. *Given V is a g -dimensional vector space, and $W \subset V^r$ a g -dimensional subspace such that the restriction of $\pi_L : V^r \rightarrow (V/L)^r$ to W is not an isomorphism for any $L \in \mathbb{P}(V)$, then $W = V_M^1$ for some $M \in \mathbb{P}(\mathbb{C}^r)$.*

Hence, if \mathcal{Z} fails to satisfy $(*)$ for $l = k - 2$, for any $s \in S$ and $z \in (\mathcal{Z}_s)_{sm}$ the tangent space $T_{\mathcal{Z}_s, z}$ must be of the form $(T_{\mathcal{A}_s})_M^1 \subset T_{\mathcal{A}_s}^{k, 0}$ for $M \in \mathbb{P}(\mathbb{C}^{k, 0})$. Here, given a vector space V we use the notation $V^{r, 0}$ for the kernel of the summation map $V^r \rightarrow V$. It follows that for each $s \in S$ we get a morphism $(\mathcal{Z}_s)_{sm} \rightarrow \mathbb{P}(\mathbb{C}^{k, 0})$. For very general s , the abelian variety \mathcal{A}_s is simple and so \mathcal{Z}_s cannot be stabilized by an abelian subvariety of \mathcal{A}_s^k . Thus, the Gauss map of \mathcal{Z}_s is generically finite on its image and so is the morphism $(\mathcal{Z}_s)_{sm} \rightarrow \mathbb{P}(\mathbb{C}^{k, 0})$. It follows that the image of this morphism must contain an open in $\mathbb{P}(\mathbb{C}^{k, 0})$. Any open in $\mathbb{P}(\mathbb{C}^{k, 0})$ contains real points and if $M \in \mathbb{P}(\mathbb{R}^{k, 0})$, then $(T_{\mathcal{A}_s})_M^1$ cannot be totally isotropic for ω_k for any non-zero $\omega \in H^0(\mathcal{A}_s, \Omega^2)$. This provides the desired contradiction. \square

The previous corollary motivates the following:

Problem 5.5. *Exhibit an abelian threefold A with $d_r(A) \geq 4$.*

We have reasons to believe that Corollary 5.3 is also not optimal. Indeed, the key obstacle to proving a stronger lower bound is the need for $(*)$ to be satisfied. A careful study of the Gauss map of cycles on very general abelian variety is likely to provide stronger results.

Similarly, we believe that Theorem 4.4 can be improved. In fact, though Conjecture 1.3 is the main conjecture of [14], it is not the most ambitious. Voisin proposes to attack Conjecture 1.3 by studying what she calls the locus Z_A of positive dimensional normalized orbits of degree k

$$Z_A := \left\{ a_1 \in A : \exists a_2, \dots, a_{k-1} : \dim \left| \{a_1\} + \dots + \{a_{k-1}\} + \left\{ - \sum_{i=1}^k a_i \right\} \right| > 0 \right\}.$$

In particular she suggests to deduce Conjecture 1.3 from the following conjecture:

Conjecture 5.6 (Voisin, Conj. 5.2 in [14]). *If A is a very general abelian variety*

$$\dim Z_A \leq k - 1.$$

Voisin shows that this conjecture implies Conjecture 1.3 but it in fact implies the following stronger conjecture:

Conjecture 5.7. *A very general abelian variety of dimension at least $k + 1$ does not have a positive dimensional orbit. i.e. $\mathcal{G}(k) \leq k + 1$.*

Remark 5.8. The previous conjecture follows from Conjecture 5.6. Indeed, if a very general abelian variety of dimension k has a positive dimensional orbit

$$|\{a_1\} + \dots + \{a_{k-1}\}|$$

of degree $k - 1$, then for any $a \in A$ the orbit

$$(5.1) \quad |\{(k-1)a\} + \{a_1 - a\} + \dots + \{a_{k-1} - a\}|$$

is positive dimensional. This was noticed by Voisin in Example 5.3 of [14]. It follows that $Z_A = A$ and so $\dim Z_A = k > k - 1$.

As mentioned above, the results of Pirola and Alzati-Pirola give $\mathcal{G}(2) \leq 3$ and $\mathcal{G}(3) \leq 4$. Our main theorem provides us with the bound $\mathcal{G}(4) \leq 6$. An interesting question is to determine if $\mathcal{G}(4) \leq 5$. This would provide additional evidence in favor of Conjecture 5.7.

APPENDIX A. SUPPORT OF ZERO-CYCLES ON ABELIAN VARIETIES

In [14] the author shows the following surprising proposition:

Proposition A.1. *Consider an abelian variety A and an effective zero-cycle $\sum_{i=1}^k \{x_i\}$ on A such that*

$$\sum_{i=1}^k \{x_i\} = k\{0_A\} \in CH_0(A).$$

Then for $i = 1, \dots, k$

$$(\{x_i\} - \{0_A\})^{*k} = 0 \in CH_0(A),$$

where $*$ denotes the Pontryagin product.

Voisin defines a subset $A_k := \{a \in A : (\{a\} - \{0\})^{*k} = 0\} \subset A$ and shows that $\dim A_k \leq k - 1$. Given a smooth projective variety X and a zero-cycle of the form $z = \sum_{i=1}^k \{x_i\} \in Z_0(X)$ the support of z is

$$\text{supp}(z) = \{x_i : i = 1, \dots, k\} \subset X.$$

Similarly we will call the k -support of z the following subset of X

$$\text{supp}_k(z) = \bigcup_{z' = \sum_{i=1}^k \{x'_i\} : z' \sim z} \text{supp}(z').$$

The previous proposition can then be rephrased as: $\text{supp}_k(k\{0_A\}) \subset A_k$. Here we present a generalization of this result. Given $\underline{x} \in X^k$ we let

$$A_{k,\underline{x}} := \{a \in A : (\{a\} - \{x_1\}) * \dots * (\{a\} - \{x_k\}) = 0 \in CH_0(A)\}.$$

One shows easily, using the same argument as Voisin, that $\dim A_{k,\underline{x}} \leq k - 1$.

Proposition A.2. *Consider an abelian variety A and effective zero-cycles $\sum_{i=1}^k \{x_i\}, \sum_{i=1}^k \{y_i\}$ on A such that*

$$\sum_{i=1}^k \{x_i\} = \sum_{i=1}^k \{y_i\} \in CH_0(A).$$

Then for $i = 1, \dots, k$

$$\prod_{j=1}^k (\{x_i\} - \{y_j\}) = 0 \in CH_0(A),$$

where the product is the Pontryagin product.

Upon presenting this result to Nori he recognized it as a more effective reformulation of results of his from around 2005. They had been obtained in an attempt to understand work of Colombo-Van Geeman [4] but left unpublished for lack of an application. We present here Nori's proof as it is more elegant than our original proof. This proof was also suggested to Voisin by Beauville in the context of Proposition A.1.

Let X be a smooth projective variety and consider the graded algebra

$$\bigoplus_{n=1}^{\infty} CH_0(X^n).$$

The multiplication is given by extending by linearity

$$\begin{aligned} X^m \times X^n &\rightarrow X^{m+n} \\ ((x_1, \dots, x_m), (x'_1, \dots, x'_n)) &\mapsto (x_1, \dots, x_m, x'_1, \dots, x'_n) \end{aligned}$$

to a product

$$Z_0(X^m) \times Z_0(X^n) \rightarrow Z_0(X^{m+n}).$$

It is easy to see that the resulting product descends to rational equivalence on the components. Let

$$R := \bigoplus_{n=1}^{\infty} CH_0(X^n)/(ab - ba) = \bigoplus_{i=1}^{\infty} R_n$$

be the abelianization of this algebra. Recall that by $CH_0(X^n)$ we mean the Chow group of zero-cycles with rational coefficients.

Lemma A.3. *If $z = \sum_{i=1}^k \{x_i\} \in Z_0(X)$ and $y \in \text{supp}_k(z)$, then*

$$(\{y\} - \{x_1\})(\{y\} - \{x_2\}) \cdots (\{y\} - \{x_k\}) = 0 \in R,$$

where the product is taken in R and we consider $\{y\} - \{x_i\}$ as elements of $R_1 \subset R$.

Proof. Since $y \in \text{supp}_k(z)$, there is $\underline{y} = (y_1 = y, y_2, \dots, y_k) \in X^k$ such that $\sum_{i=1}^k \{y_i\} = \sum_{i=1}^k \{x_i\}$. Consider the diagonal embeddings

$$\Delta_l : X \rightarrow X^l.$$

These give linear maps

$$\Delta_{l*} : R_1 \rightarrow R_l$$

such that

$$\Delta_{l*} \left(\sum_{i=1}^k \{y_i\} \right) = \sum_{i=1}^k \{y_i\}^l \in R_l.$$

Since

$$\sum_{i=1}^k \{y_i\} = \sum_{i=1}^k \{x_i\}$$

we get

$$p_l(\underline{y}) = \sum_{i=1}^k \{y_i\}^l = \sum_{i=1}^k \{x_i\}^l = p_l(\underline{x}) \in R_l,$$

where p_l is the l^{th} Newton polynomial and $\underline{x} = (\{x_1\}, \dots, \{x_k\})$. On the other hand we have

$$(\{y\} - \{x_1\})(\{y\} - \{x_2\}) \cdots (\{y\} - \{x_k\}) = \{y\}^k - e_1(\underline{x})\{y\}^{k-1} + \cdots + (-1)^k e_k(\underline{x}) \in R_k,$$

where e_l is the l^{th} elementary symmetric polynomial. Since the elementary symmetric polynomials can be written as polynomials in the Newton polynomials and since $p_l(\underline{y}) = p_l(\underline{x})$ for all $l \in \mathbb{N}$, we get

$$\prod_{i=1}^k (\{y\} - \{x_i\}) = \sum_{i=0}^k (-1)^i \{y\}^{k-i} e_i(\underline{x}) = \sum_{i=0}^k (-1)^i \{y\}^{k-i} e_i(\underline{y}) = \prod_{i=1}^k (\{y\} - \{y_i\}) = 0 \in R_k.$$

□

Proof of Proposition A.2 (Nori). If $X = A$ is an abelian variety we have a summation morphism $A^l \rightarrow A$ inducing maps

$$CH_0(A^l) \rightarrow CH_0(A),$$

and so a map

$$\sigma : R \rightarrow CH_0(A)$$

such that

$$\sigma \left(\prod_{i=1}^k (\{y\} - \{x_i\}) \right) = (\{y\} - \{x_1\}) * \dots * (\{y\} - \{x_k\}) \in CH_0(A).$$

□

Lemma A.3 in fact has many more interesting corollaries. Consider $p(t_1, \dots, t_k) \in \mathbb{C}[t_1, \dots, t_k]$ and the S_k action on $\mathbb{C}[t_1, \dots, t_k]$ given by permutation of the variables. Let $H_p \subset S_k$ be the subgroup stabilizing p .

Corollary A.4. *Consider an abelian variety A and effective zero-cycles $\sum_{i=1}^k \{x_i\}, \sum_{i=1}^k \{y_i\}$ on A . Then*

$$\sum_{i=1}^k \{x_i\} = \sum_{i=1}^k \{y_i\} \in CH_0(A)$$

if and only if

$$\prod_{\sigma \in S_k/H_p} (p(\{y_1\}, \dots, \{y_k\}) - (\sigma \cdot p)(\{x_1\}, \dots, \{x_k\})) = 0 \in CH_0(A)$$

for every $p \in \mathbb{C}[t_1, \dots, t_k]$. Here the product is the Pontryagin product.

Proof. The if direction follows trivially from considering $p(t_1, \dots, t_k) = t_1 + \dots + t_k$. The proof of Lemma A.3 completes the argument. □

The special case $p = t_1$ is Proposition A.2. Another corollary of Lemma A.3 is the following:

Corollary A.5 (Nori). *Given an effective zero-cycle $z = \sum_{i=1}^k \{x_i\}$ on an abelian variety A , and $y_1, \dots, y_{k+1} \in \text{supp}_k(z)$, the following identity is satisfied*

$$\prod_{i < j} (\{y_i\} - \{y_j\}) = 0 \in CH_0(A).$$

Proof. Let e_l be the l^{th} elementary symmetric polynomial. By Lemma A.3 we have

$$\{y_i\}^k - e_1(\{\underline{x}\})\{y_i\}^{k-1} + \dots + (-1)^k e_k(\{\underline{x}\}) = 0 \in R_k$$

for $i = 1, \dots, k+1$, where $\{\underline{x}\} = (\{x_1\}, \dots, \{x_k\})$. This gives a non-trivial linear relation between the rows of the Vandermonde matrix $(\{y_i\}^{j-1})_{1 \leq i, j \leq k+1}$. It follows that the Vandermonde determinant vanishes. Using the morphism $\sigma : R \rightarrow CH_0(A)$ from the proof of A.2 finishes the argument. □

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